A problem coming from group theory

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Original Problem

$$\begin{matrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} & \cdots \\ a_{2,1} & \cdots & a_{2,n-1} & a_{2,n} & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & \cdots & \cdots & a_{n,n} & \cdots \end{matrix}$$

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- $a_{i,j} \in B_{i,j}$.

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- (row sets distinct at any point) $\{a_{i,1}, \cdots, a_{i,k}\} \neq \{a_{j,1}, \cdots, a_{j,k}\}$
- Easy when all $B_{i,j} = [n]$. Try!

Example

When $n = 5$, all $B_{i,j} = \{1, 2, 3, 4, 5\}$

$$\begin{matrix} 1 & 2 & 3 & 4 & 4 & 4 & \cdots \\ 2 & 3 & 4 & 5 & 5 & 5 & \cdots \\ 3 & 4 & 5 & 1 & 1 & 1 & \cdots \\ 4 & 5 & 1 & 2 & 2 & 2 & \cdots \\ 5 & 1 & 2 & 3 & 3 & 3 & \cdots \end{matrix}$$

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- Observe : First $n - 1$ columns and the columns after behave slightly differently.
- If all $B_{i,j}$ are different, also easy!

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- If all $B_{i,j}$ are different, also easy!
- Harder for other cases!
- Number of columns? Can we just think of finite cases?
- Konig Lemma : G a connected graph with infinitely many vertices, where degree is finite, G contains an infinitely long simple path.

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- (Curtin O.) : Yup! Problem is true for any n.

Generalization

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- (first n-1 columns) $a_{i,1}, \cdots, a_{i,n-1}$ are mutually distinct.
- (*, afterwards) $a_{i,t} \notin \{a_{i,1}, \cdots, a_{i,n-2}\}$.
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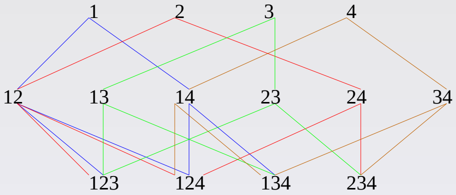
- Ex : $1234|567?, B = \{1,2,3,4,8,9\}$ we choose 9, row set is now $\{1,2,3,4,5,6,7,9\}$.

System of distinct representative of n binary posets



- Growth of n-sets, where you are offered two options each time!

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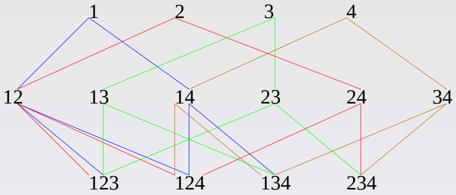

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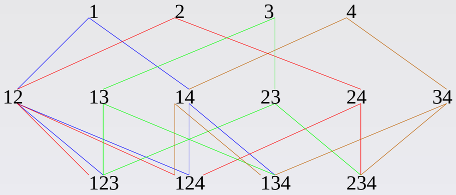

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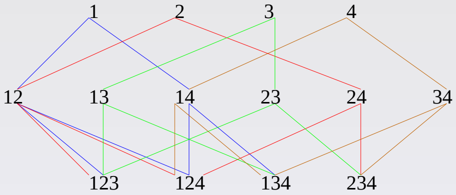

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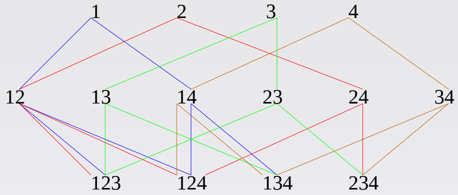

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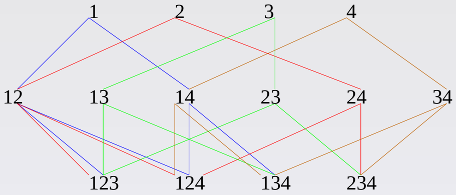

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- Can you find a system of distinct representative for the given n hypergraphs?

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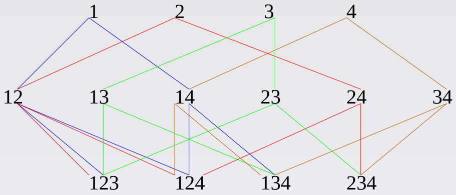

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- Can you find a chain for each binary poset so that the chains are pairwise disjoint?
- $(n-1)$-partite graph. Hypergraph where edges are chains along the binary posets.
- Can you find a system of distinct representative for the given n hypergraphs?
- (Aharoni) Hall's theorem for hypergraphs... Not easy to use.

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- Ex : $A_1 = \{1,2,3\}, A_2 = \{2,3,4\}, A_3 = \{3,4,1\}, A_4 = \{4,1,2\}$. Done!

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- For each set, you must pick the correct element among the 2 choices so that all A_i's are still mutually distinct!
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- Ex : $A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, A_4 = \{4\}$. Adversary offers $\{2,3\}, \{3,4\}, \{4,1\}, \{1,2\}$ each.
- Ex : $A_1 = \{1,2\}, A_2 = \{2,3\}, A_3 = \{3,4\}, A_4 = \{4,1\}$. Adversary offers $\{3,4\}, \{4,1\}, \{1,2\}, \{2,3\}$ each.
- Ex : $A_1 = \{1,2,3\}, A_2 = \{2,3,4\}, A_3 = \{3,4,1\}, A_4 = \{4,1,2\}$. Done!
- Can't be too crowded! Ex : $12345, 12346, 12347, 12356, 12367$, offered $67, 57, 56, 47, 45$.

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- But still not enough..