Minimal presentations of shifted numerical monoids

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To shift a numerical monoid...

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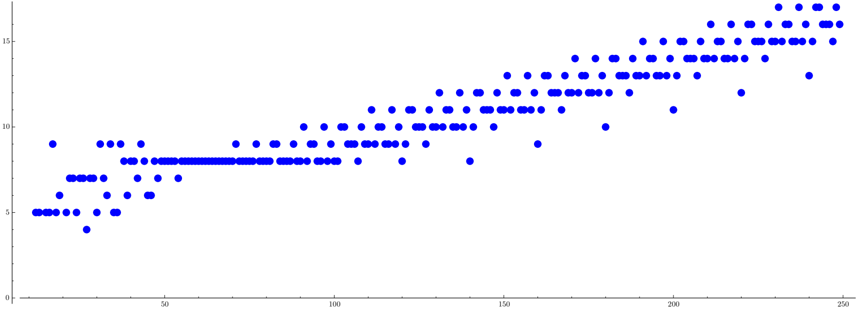

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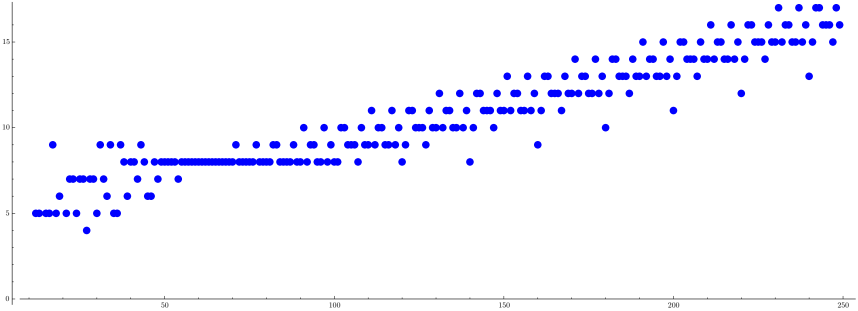

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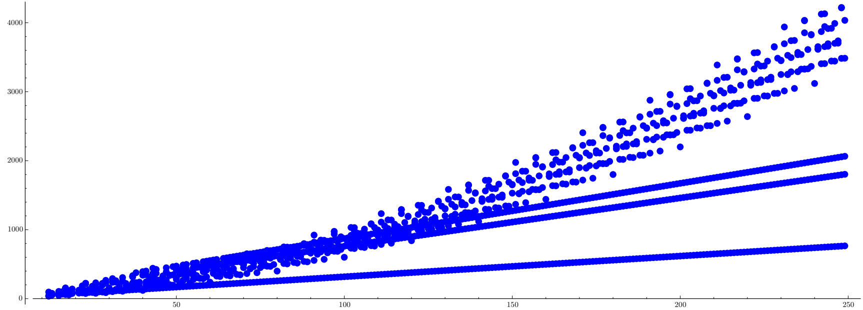
$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$: Graded degrees for $\beta_0(M_n)$



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Underlying cause: minimal presentations!

Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

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that is closed under *translation*.

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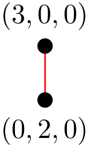

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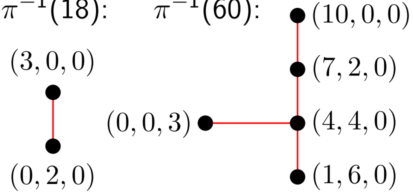

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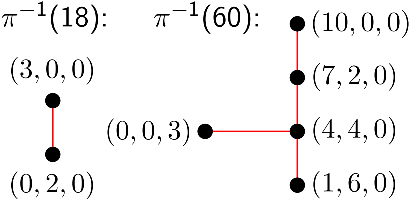

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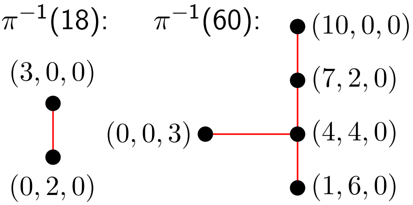

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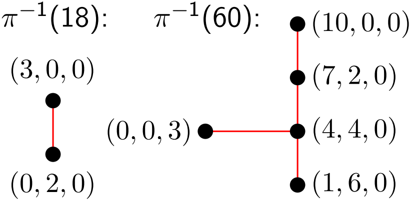

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$$I_S = \langle x^u - x^v : (u,v) \in \rho \rangle$$

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$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

The shifting map

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Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{matrix} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{matrix} \right\}$$

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M_{470}:

$$\left\{ \begin{matrix} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)), ((\ \ 0,1,6,0),(4,0,0,\ \ 3)), ((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((21,5,0,0),(0,0,0,25)), ((26,1,0,0),(0,0,4,22)), ((27,0,0,0),(0,2,2,22)) \end{matrix} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

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M_{490}:

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- Only missing link: transitivity.
  Φ_n only preserves *monotone* chain connectivity.

The main result

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$*, the image* $\Phi_n(\ker \pi_n)$ *generates* $\ker \pi_{n+r_k}$ *as a congruence.*

$$\{\textit{minimal presentations of } M_n\} \quad \leftrightsquigarrow \quad \{\textit{minimal presentations of } M_{n+r_k}\}$$
$$\rho \subset \ker \pi_n \quad \longmapsto \quad \Phi_n(\rho) \subset \ker \pi_{n+r_k}$$

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Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually r_k-periodic:

  Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$

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  $\Delta(M_n) = \{d\}$ when $||\mathbf{a}| - |\mathbf{a}'|| \in \{0, d\}$ for all $(\mathbf{a}, \mathbf{a}') \in \rho$

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- The function $n \mapsto \mathrm{c}(M_n)$ is eventually r_k-quasilinear:

  $\mathrm{c}(M_n)$ is determined by $\{$minimal presentations of $M_n\}$

Application: computing minimal presentations

Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

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Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

For any $n > r_k^2$, *the image* $\Phi_n(\ker \pi_n)$ *generates* $\ker \pi_{n+r_k}$ *as a congruence.*

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$S = \langle 1234, 1240, 1243, 1254 \rangle$

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$S = \langle 1234, 1240, 1243, 1254 \rangle = \langle n, n + r_1, \ldots, n + r_k \rangle$

$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

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Verify $n > r_k^2$:

Application: computing minimal presentations

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Verify $n > r_k^2$: $1234 > 400$

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$$n = 1234 \qquad r_3 = 1254 - 1234 = 20$$

Verify $n > r_k^2$: $1234 > 400$ ✓

$\langle 414, 420, 423, 434 \rangle$:

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Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

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$((0,1,6,0),(4,0,0,3))$,
$((0,3,0,0),(1,0,2,0))$,
$((21,1,0,0),(0,0,0,21))$,
$((25,0,0,0),(0,0,6,18))$

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$((21,1,0,0),(0,0,0,21))$,		$((62,1,0,0),(0,0,0,62))$,
$((25,0,0,0),(0,0,6,18))$		$((66,0,0,0),(0,0,6,59))$

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n	M_n	Min. Pres. Runtime
50	$\langle 50, 56, 59, 70 \rangle$	1 ms
200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min
5000	$\langle 5000, 5006, 5009, 5020 \rangle$	18 min
10000	$\langle 10000, 10006, 10009, 10020 \rangle$	4.2 hr

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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

Future work

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Frobenius number: $F(S) = \max(\mathbb{N} \setminus S)$.

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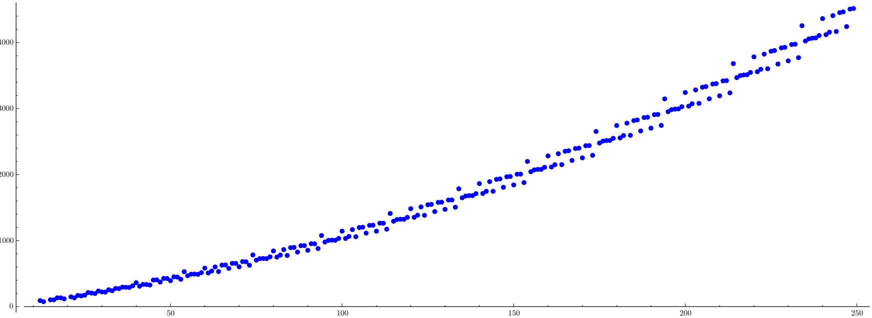

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Thanks!