A signed structure theory for oriented hypergraphs

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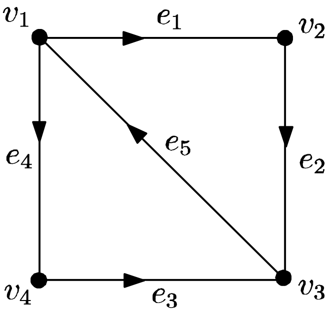
Incidence vs Adjacency vs Edge

1. In a graph: Incidence $\Rightarrow$ adjacency $\Leftrightarrow$ edge (sign $+$ is implied).
    1. These separate in an oriented hypergraph.
2. Incidence Matrix Magic: Generalizing the cycle space.
3. OH Matrices and Unifying Entries.
4. Weak Walk Covers and the Matrix-tree Theorem.

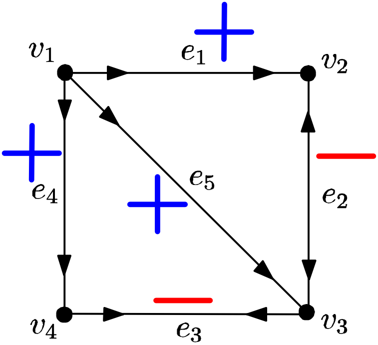
Incidence Matrix: Graphs



$$\Leftrightarrow \mathrm{H}_G = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

- Minimal Dependency $\mathrm{H} \Longleftrightarrow$ Circle in G.

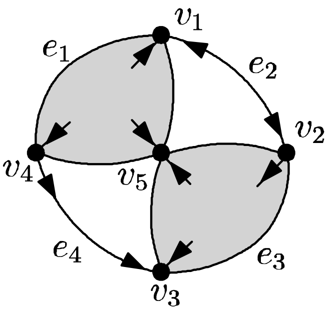
Incidence Matrix: Signed Graphs



$$\Leftrightarrow \mathrm{H}_G = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

- Minimal Dependency $\mathrm{H} \iff$ Positive circle or Contrabalanced handcuff in G.

Incidence Matrix: Oriented Hypergraphs



$$\Leftrightarrow \mathrm{H}_G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- Minimal Dependency $\mathrm{H} \Longleftrightarrow$ Balanced subdivision of balanced hypercircles (*balanced*), Camion connections of disjoint floral families (*balanceable*), or ??? (*unbalanceable*).

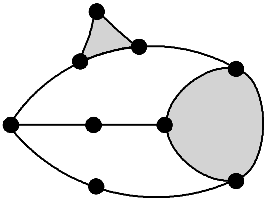
Balanceability: Measuring Negative Circles

Definitions

Type	Condition	Note
Balanced	No negative circles.	All graphs.
Balanceable	Incidence reversals result in balance.	All signed graphs.
Unbalanceable	Not balanceable.	No signed graphs.

Theorem

The only obstruction to balanceability is three internally-disjoint paths that begin at an edge and terminate at a vertex.

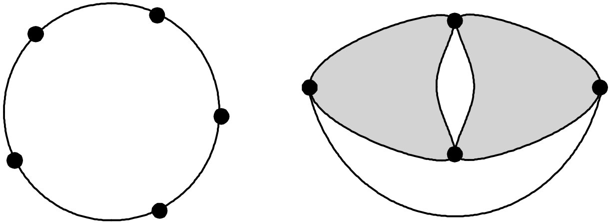


Cross-theta

Hypergraphic Circle Analogs - Flowers

Definition (Flower)

A *flower* is a minimal inseparable oriented hypergraph.

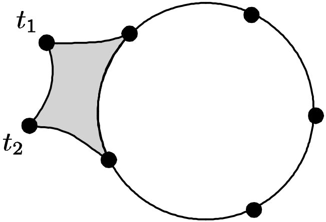


F_1 F_2

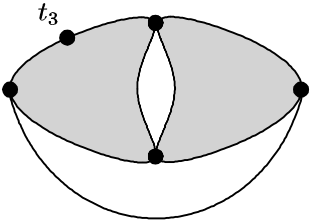
The Pseudo-flower Problem

Definition (Pseudo-flower)

A *pseudo-flower* is an OH where the weak-deletion of thorns results in a flower.



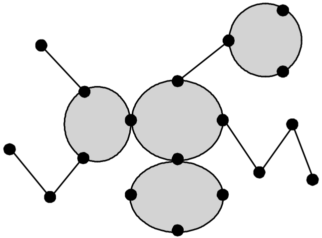
P_1



P_2

Definition

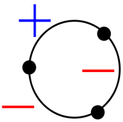
An *artery* is a subdivision of an edge.

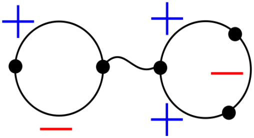


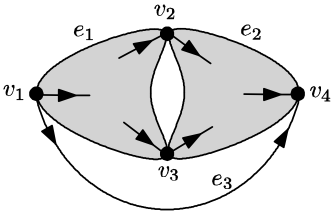
Theorem (R. 2013)

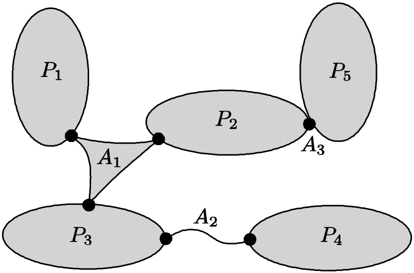
The only balanced minimal dependencies are balanced flowers or arterial connections of balanced pseudo-flowers. (* Up to balanced subdivision and 2-vertex-contraction.)*

Some Minimal Dependencies





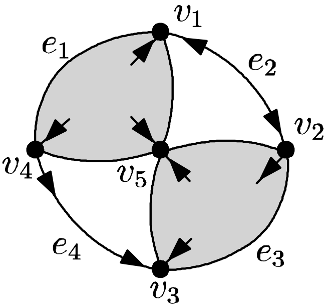




Oriented Hypergraphic Matrices

Oriented Hypergraphic Matrices

- Incidence Matrix: H_G
- Degree Matrix: D_G
- Adjacency Matrix: A_G
- **Laplacian Matrix:** $L_G := D_G - A_G = \mathrm{H}_G \mathrm{H}_G^T$



$$\Leftrightarrow L_G = \begin{bmatrix} 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 1 \\ 1 & 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}$$

Universal Theorems

Definition

A *weak walk* is a sequence $\widetilde{w} = a_0, i_1, a_1, i_2, a_2, i_3, a_3, \ldots, a_{n-1}, i_n, a_n$ of vertices, edges and incidences, where $\{a_k\}$ is an alternating sequence of vertices and edges, and i_h is an incidence containing a_{h-1} and a_h.

Theorem (Chen, Rao, R. and Yang. 2015)

$\left(A_G^k\right)_{ij} = w^{\pm}(v_i, v_j; k)$.

$\left(L_G^k\right)_{ij} = (-1)^k \cdot \widetilde{w}^{\pm}(v_i, v_j; k)$.

Unifying Entries

Theorem

Let G be an oriented hypergraph.

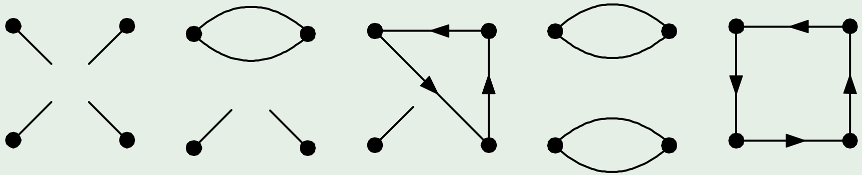
1. H_G *is the half-walk matrix.*
2. D_G *is the strictly* 1*-weak walk matrix. Called* <u>*backsteps*</u>*.*
3. A_G *is the* 1*-(non-weak)-walk matrix.*
4. L_G *is negative the* 1*-weak-walk matrix.*

Weak Walk Covers

Stirling Covers

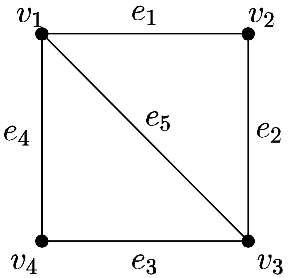
Example

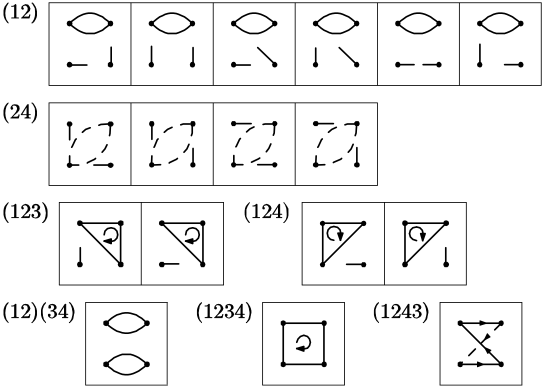
Representations of some permutations via Stirling covers.



- When do they exist in a graph?
- Include any "missing" adjacencies/backsteps. Consider their sign to be 0.
- A *weak walk contributor of G* is a labeling of a Stirling cover.

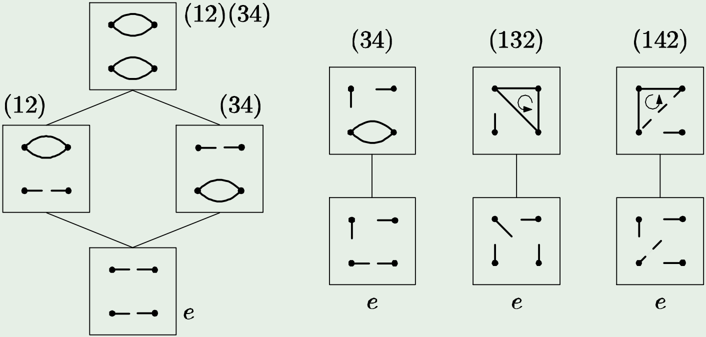
Some Contributors





Activation Classes

Example



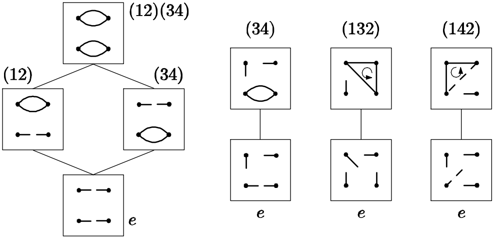
Definition

The *sign of a contributor c* is defined by

$$sgn(c) = (-1)^{pc(c)}(0)^{zc(c)}.$$

Theorem

Given a graph G, for every contribution class C we have $\sum_{c \in C} sgn(c) = 0$.

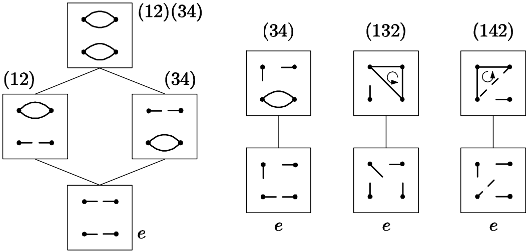


Contributors are Cycle Cover Analogs

Theorem

For an oriented graph G,

$$\det(L_G) = \sum_{c \in \mathfrak{C}} sgn(c) = 0$$

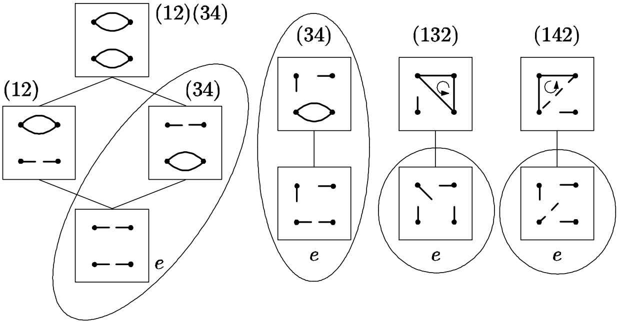


The Matrix-tree Theorem

Cutting the Contributor Posets

Definitions

$M_0(v_r, v_r; C)$	Maximal element(s) in C where v_r is in no active circle.
$m_1(v_r, v_k; C)$	Minimal element in class C where v_r is in an active circle.



Relative Signings

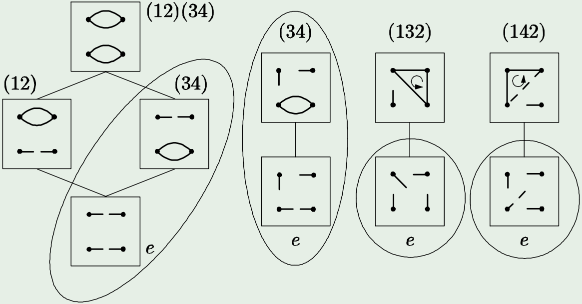
Definitions

1. Let $\underline{\underline{(c;\mathbf{v};\mathbf{w})}}$ be contributor c with each adjacency or non-zero backstep $v_{r_i} \longrightarrow v_{k_i}$ removed (zero backsteps are not removed).
2. The $\underline{\underline{\textit{sign of a contributor } c \textit{ with respect to } (\mathbf{v};\mathbf{w})}}$ is defined as

$$sgn(c;\mathbf{v};\mathbf{w}) = (-1)^{pc(c;\mathbf{v};\mathbf{w})}(0)^{zc(c;\mathbf{v};\mathbf{w})}(-1)^{pl(c;\mathbf{v};\mathbf{w})}(0)^{zl(c;\mathbf{v};\mathbf{w})}$$

3. Define $\underline{\underline{\mathfrak{C}^{=}_{\neq 0}(\mathbf{v};\mathbf{w})}}$ as the set of all non-zero contributors in the classes that all have the same sign within a single class.
4. $\underline{\underline{\mathfrak{C}^{\times}(\mathbf{v};\mathbf{w})}} := \{(c;\mathbf{v};\mathbf{w}) | c \in \mathfrak{C}^{=}_{\emptyset}(\mathbf{v};\mathbf{w})$ (identification of contributors along the admissible exceptions).

Example (A v_1-cut)



The only members in a $\mathfrak{C}^{=}_{\neq 0}(v_1; v_k)$ are both contributors of the third class. The top member of the fourth class is not a member of $\mathfrak{C}^{=}_{\neq 0}(v_1; v_4)$, but is a member of $\mathfrak{C}^{=}_{\neq 0}(v_4; v_2)$ since the adjacency $v_4 \longrightarrow v_2$ does not exist in G.

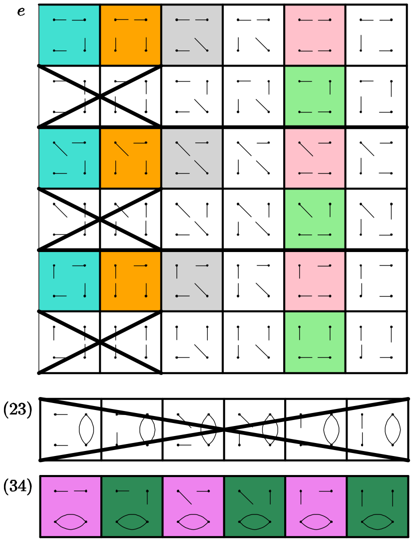
The Matrix-tree Theorem

Theorem

The number of spanning trees in a graph G, $T(G)$, is:

$$T(G) = \varepsilon(v;w) \sum_{c \in \mathfrak{C}^{\times}(v;w)} sgn(c;v;w).$$

The Matrix-tree Theorem



All Minors Matrix-tree Theorem

Theorem

Let $(L_G; \mathbf{v}; \mathbf{w})$ *be the minor determined by removing the rows of* $\mathbf{v}$ *and the columns of* $\mathbf{w}$*, then*

$$\det(L_G; \mathbf{v}; \mathbf{w}) = \varepsilon(\mathbf{v}; \mathbf{w}) \sum_{c \in \mathfrak{C}^{\times}(\mathbf{v}; \mathbf{w})} sgn(c; \mathbf{v}; \mathbf{w})$$

The Signed Graph AMMTT

Theorem

Let $(L_G; \mathbf{v}; \mathbf{w})$ be the minor determined by removing the rows of $\mathbf{v}$ and the columns of $\mathbf{w}$, then

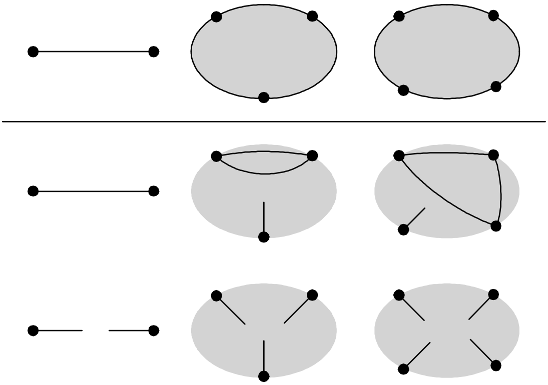
$$\det(L_G; \mathbf{v}; \mathbf{w}) = \varepsilon(\mathbf{v}; \mathbf{w}) \sum_{c \in \mathfrak{C}^{\times}(\mathbf{v};\mathbf{w})} sgn(c; \mathbf{v}; \mathbf{w})$$

Cracking Hypergraphs

Hyperedges and Stirling covers

Definition

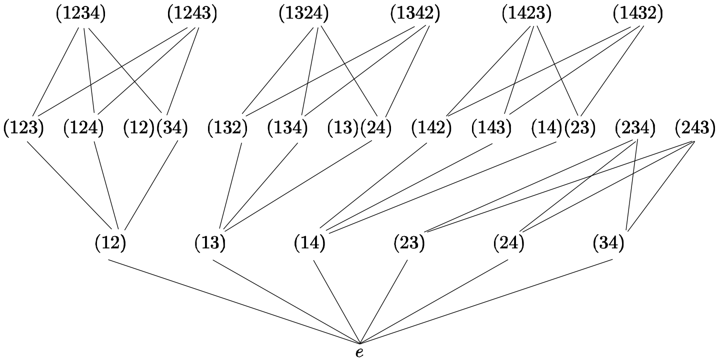
A closed vertex-cotrail is called a *cirque*.



Cirque Augmentation

The Cirque Order and Extending Cuts

- Multiple $M_0(v_r, v_r; C)$ and $m_1(v_r, v_k; C)$ elements.



All Minors Matrix-tree Theorem

Theorem

Let $(L_G; \mathbf{v}; \mathbf{w})$ *be the minor determined by removing the rows of* $\mathbf{v}$ *and the columns of* $\mathbf{w}$, *then*

$$\det(L_G; \mathbf{v}; \mathbf{w}) = \varepsilon(\mathbf{v}; \mathbf{w}) \sum_{c \in \mathfrak{C}^{\times}(\mathbf{v};\mathbf{w})} sgn(c; \mathbf{v}; \mathbf{w})$$

- Combine the cirque order by augmentation and activation order. (Cirque order first.)

The End!