STABILITY OF NONSURJECTIVE ε-ISOMETRIES OF BANACH SPACES

LIXIN CHENG

Given $\varepsilon > 0$, a mapping f from a Banach space X to a Banach space Y is said to be an ε-isometry provided

$$(1) \qquad |\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon, \quad \text{for all } x, y \in X.$$

If $\varepsilon = 0$, then it is simply called an isometry. Historically speaking, the study of ε-isometry divided into the following four cases: (i) $\varepsilon = 0$ and f is surjective; (ii) $\varepsilon = 0$ and f is nonsurjective; (iii) $\varepsilon \neq 0$ and f is surjective and (iv) $\varepsilon \neq 0$ and f is nonsurjective. The first celebrated result (for Case (i)) due to S. Mazur and S. M. Ulam is: Every surjective isometry $f : X \to Y$ is an affine isometry. The most remarkable result about nonsurjective isometry by T. Figiel is: For every isometry $f : X \to Y$ with $f(0) = 0$, there exists a linear operator $P : L(f) \equiv \overline{\text{span}} f(X) \to X$ with $\|P\| = 1$ such that $P \circ f = I$ (the identity) on X. For (surjective) isometry, a question proposed by D.H. Hyers and S.M.Ulam in 1945 is: whether for every surjective ε-isometry $f : X \to Y$ with $f(0) = 0$ there exists a bijective linear isometry $U : X \to Y$ and $\gamma > 0$ such that

$$(2) \qquad \|f(x) - Ux\| \leq \gamma\varepsilon, \quad \text{for all } x \in X.$$

After many years efforts of a number of mathematicians, the following sharp result was finally obtained by M. Omladič and P. Šemrl: If $f : X \to Y$ is a surjective ε-isometry with $f(0) = 0$, then there is a bijective linear isometry $U : X \to Y$ such that

$$(3) \qquad \|f(x) - Ux\| \leq 2\varepsilon, \quad \text{for all } x \in X.$$

For nonsurjective ε-isometry, S. Qian proposed the following problem in 1995, and meanwhile, he showed that the answer to this question is affirmative if both X and Y are L_p-spaces.

Whether there exists a constant $\gamma > 0$ depending only on X and Y with the following property: For each into ε-isometry $f : X \to Y$ with $f(0) = 0$ there is a bounded linear operator $U : L(f) \to X$ such that

$$(4) \qquad \|Uf(x) - x\| \leq \gamma\varepsilon \ \text{ for all } x \in X.$$

1991 *Mathematics Subject Classification.* Primary 46B04, 46B20, 47A58; Secondary 26E25, 46A20, 26A24.

Key words and phrases. ε-isometry, nonlinear operator, set-valued mapping, Banach space.

support partially by the Natural Science Foundation of China, grant 11071201.

But unfortunately, in the same paper, he gave the following counterexample showing that for any separable Banach space with a uncomplemented subspace, the answer to this problem is always negative:

Let $\varepsilon > 0$, and let X be an uncomplemented subspace of some separable Banach space Y. Let f_0 be a bijective mapping from X to $B(Y)$ with $f_0(0) = 0$. Define $f : X \to Y$ by $f(x) = x + \varepsilon f_0(x)/2$ for all $x \in X$. Then f is an ε-isometry with $L(f) = Y$. But there never be U and γ satisfying (4).

For an ε isometry f with $f(0) = 0$, let E be the maximal subspace contained in $\overline{\mathrm{co}}\{f(X) \cup -f(X)\}$. Note that we have $E = X$ in Qian's counterexample. We can see that the assumption of E being complemented in Y is essential.

In this talk, we show the following theorem.

Theorem 0.1. *Let X and Y be Banach spaces, and let $f : X \to Y$ be an ε-isometry for some $\varepsilon \geq 0$ with $f(0) = 0$. Then*

(i) For every $x^* \in X^*$, there exists $\phi_{x^*} \in Y^*$ with $\|\phi_{x^*}\| = \|x^*\| \equiv r$ such that

$$|\langle \phi_{x^*}, f(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon r, \text{ for all } x \in X. \tag{5}$$

(ii) If Y is reflexive and if E is α-complemented in Y, then there is a bounded linear operator $T : Y \to X$ with $\|T\| \leq \alpha$ such that

$$\|Tf(x) - x\| \leq 4\varepsilon, \quad \text{for all } x \in X. \tag{6}$$

(iii) If Y is reflexive, smooth and locally uniformly convex, and if E is α-complemented in Y, then there is a bounded linear operator $T : Y \to X$ with $\|T\| \leq \alpha$ such that

$$\|Tf(x) - x\| \leq 2\varepsilon, \quad \text{for all } x \in X. \tag{7}$$

Lixin Cheng: School of Mathematical Sciences, Xiamen University, Xiamen, 361005, China

E-mail address: lxcheng@xmu.edu.cn