Spring 2006 Math 151
Exam 3B: Solutions
Mon, 01/May ©2006, Art Belmonte

1. (b) Given $f(x) = \sin^{-1}(x^2)$, we have

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x \\ f'\left(\frac{1}{\sqrt{2}}\right) &= \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{6}}. \end{aligned}$$

2. (b) Apply L'Hospital's Rule and algebra.

$$\begin{aligned} \lim_{\theta\to 0} \frac{\tan^{-1}\theta - \theta}{\theta^3} &\overset{\text{L'H}}{=} \lim_{\theta\to 0} \frac{\frac{1}{1+\theta^2} - 1}{3\theta^2} \\ &= \lim_{\theta\to 0} \frac{\frac{-\theta^2}{1+\theta^2}}{3\theta^2} \\ &= \lim_{\theta\to 0} \frac{-1}{3(1+\theta^2)} \\ &= -\frac{1}{3} \end{aligned}$$

3. (b) Given $g(x) = \ln\left(1 + e^{\sqrt{x}}\right)$, we have

$$\begin{aligned} g'(x) &= \frac{1}{1+e^{x^{1/2}}} \cdot e^{x^{1/2}} \cdot \frac{1}{2}x^{-1/2} \\ &= \frac{e^{\sqrt{x}}}{2\sqrt{x}\left(1+e^{\sqrt{x}}\right)}. \end{aligned}$$

4. (e) The limit $\lim_{x\to\infty} (1+2x)^{1/x}$ involves an indeterminate power, ∞^0. We use the four-step procedure from Section 4.8.

   - Let $y = (1+2x)^{1/x}$.
   - Then $\ln y = \dfrac{\ln(1+2x)}{x}$, an indeterminate quotient ∞/∞ as $x \to \infty$.
   - Therefore, $\lim_{x\to\infty} \dfrac{\ln(1+2x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \dfrac{\frac{2}{1+2x}}{1} = 0$.
   - Hence, $\lim_{x\to\infty} (1+2x)^{1/x} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$.

5. (b) Let $f(x) = \frac{1}{3}x^3 - \frac{7}{2}x^2 + 6x + 9$. Then

$$f'(x) = x^2 - 7x + 6 \quad = \quad (x-1)(x-6)$$

   is positive for $x > 6$ or $x < 1$ and negative for $1 < x < 6$. Therefore, f is decreasing for $1 < x < 6$.

6. (c) With f as in #5, $f''(x) = 2x - 7 > 0$ for $x > \frac{7}{2}$. This is where f is concave up.

7. (b) From #5, $f'(x) = (x-1)(x-6) = 0$ provided $x = 0$ or $x = 6$.

   - As x increases through 6, $f'(x) = (x-1)(x-6)$ changes from $-$ to 0 to $+$. By the First Derivative Test, a local minimum occurs at $x = 6$. As x increases through 1, $f'(x)$ changes from $+$ to 0 to $-$. This signifies a local maximum.
   - Alternatively, use the Second Derivative Test: $f''(6) = 5 > 0$ signifies a local minimum, whereas $f''(1) = -5 < 0$ indicates a local maximum.

8. (a) Note that $\frac{d}{dx}(x\ln x - x) = (1)\ln x + x\left(\frac{1}{x}\right) - 1 = \ln x$. Hence $x\ln x - x$ is an antiderivative of $\ln x$.

9. (d) Let $f(x) = 2x^3 - 3x^2 - 12x + 5$. Use the Extreme Value Theorem.

$$\begin{aligned} f'(x) = 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \implies x = -1, 2 \\ x &= 2 \in [0,4] \end{aligned}$$

x	$f(x)$
0	5
-1	12
2	-15
4	37

   The absolute value of f on $[-2,4]$ is $f(4) = 37$.

10. (a) Given $y(t) = A + e^{kt}$ with $y(2) = 4$ and $y(4) = 6$, we substitute to obtain

$$\begin{aligned} 6 &= A + e^{4k} \\ 4 &= A + e^{2k} \end{aligned}$$

   Subtraction gives $2 = e^{4k} - e^{2k} = \left(e^{2k}\right)^2 - e^{2k} = w^2 - w$. Thus $0 = w^2 - w - 2 = (w+1)(w-2)$ whence $0 < e^{2k} = w = -1, 2$. Therefore $e^{2k} = 2$, from which $2k = \ln 2$ or $k = \frac{1}{2}\ln 2$.

11. (c) We have $\sum_{i=3}^{6} (2i-1) = 5 + 7 + 9 + 11 = 32$.

12. (a) The statement "$x = 2$ is a relative minimum" is false. If it were true, then by the First Derivative Test f'would change sign from $-$ to $+$ as x increases through 2. From the graph of f' on the exam, however, we see that f' is positive on either side of $x = 2$.

13. Use logarithmic differentiation.

$$\begin{aligned} y &= (3x + \ln x)^x \\ \ln y &= x\ln(3x + \ln x) \\ \frac{1}{y}\frac{dy}{dx} &= (1)\ln(3x+\ln x) + x\left(\frac{1}{3x+\ln x}\right)\left(3 + \frac{1}{x}\right) \\ \frac{dy}{dx} &= (3x+\ln x)^x\left(\ln(3x+\ln x) + \frac{3x+1}{3x+\ln x}\right) \end{aligned}$$

14. Recall that velocity is the first derivative of position and that acceleration is the second derivative of position.

Antidifferentiate and resolve constants along the way.

$$\begin{aligned} a = s'' &= 6t + 2 \\ v = s' &= 3t^2 + 2t + C \\ 8 = v(3) &= 27 + 6 + C \\ C &= -25 \\ v = s' &= 3t^2 + 2t - 25 \\ s &= t^3 + t^2 - 25t + K \\ 9 = s(0) &= K \\ s &= t^3 + t^2 - 25t + 9 \end{aligned}$$

15. Since the population y increases exponentially with time, we use the law of exponential growth.

$$\begin{aligned} y &= y_0 e^{kt} \\ 1.02 y_0 &= y_0 e^{10k} \\ 1.02 &= e^{10k} \\ \ln 1.02 &= 10k \\ k &= \tfrac{1}{10} \ln 1.02 \\ y &= y_0 e^{\left(\frac{1}{10}\ln 1.02\right)t} = y_0 (1.02)^{t/10} \\ 2y_0 &= y_0 (1.02)^{t/10} \\ 2 &= (1.02)^{t/10} \\ \ln 2 &= \tfrac{t}{10} \ln 1.02 \\ t &= \frac{10 \ln 2}{\ln 1.02} \approx 350 \text{ min} \end{aligned}$$

16.
- From the exam diagram, the area of the inscribed triangle is $A = \frac{1}{2}BH = \frac{1}{2}(2x)\left(16 - x^2\right) = 16x - x^3$, $0 \le x \le 4$ (allowing for degenerate triangles so that we may employ the Extreme Value Theorem).
- Now $A' = 16 - 3x^2 = 0$ implies $x = \pm 4/\sqrt{3}$. Only $x = 4/\sqrt{3}$ is in the domain of A. Since $A(0) = 0$ and $A(3) = 0$, we conclude that the maximum area is $A\left(4/\sqrt{3}\right) = \frac{128}{9}\sqrt{3}$.
- [Alternatively, take the domain of A to be $0 < x < 4$. Since $A'' = -6x < 0$ on this domain, A is concave down everywhere. Therefore, the relative maximum that occurs at $x = 4/\sqrt{3}$ is actually an absolute maximum.]

17. (a)
- We recognize $\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^5$ to be the limit of right Riemann sums for the function $f(x) = (1+x)^5$ on the interval [0, 3]. (Note that $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $x_i^* = 0 + i\,\Delta x = \frac{3i}{n}$.)
- Hence this limit represents the definite integral $\int_0^3 (1+x)^5\, dx$.

(b) Use the properties of definite integrals.

$$\begin{aligned} &\int_{-1}^{5} f(x)\, dx - \int_{-1}^{0} f(x)\, dx + \int_{5}^{8} f(x)\, dx \\ = &\int_{-1}^{8} f(x)\, dx + \int_{0}^{-1} f(x)\, dx \\ = &\int_{0}^{8} f(x)\, dx \end{aligned}$$

18. The step size is $\Delta x = \dfrac{b-a}{n} = \dfrac{10-0}{5} = 2$. Therefore, the right Riemann sum is

$$\sum_{i=1}^{5} f(x_i)\,\Delta x = 2(3 - 2 - 1 + 2 + 4) = 2(6) = 12.$$