Fall 2006 Math 151
Exam 1A: Solutions
Mon, 02/Oct ©2006, Art Belmonte

1. (a) We have

$$\begin{aligned} & \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \\ = & \lim_{h\to 0} \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)} \\ = & \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6}. \end{aligned}$$

2. (e) Use the definition and properties of the dot product.

$$\begin{aligned} & \mathbf{v}\cdot(2\mathbf{w}-3\mathbf{v}) \\ = & 2\mathbf{v}\cdot\mathbf{w}-3\mathbf{v}\cdot\mathbf{v} \\ = & 2\,|\mathbf{v}|\,|\mathbf{w}|\cos 60^\circ - 3\,|\mathbf{v}|^2 \\ = & 2\,(2)\,(3)\left(\tfrac{1}{2}\right) - 3\,(2)^2 \\ = & 6-12=-6 \end{aligned}$$

3. (b) With $\mathbf{a}=[2,-3]$ and $\mathbf{b}=[1,2]$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\begin{aligned} \text{comp}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right) \\ &= \left(\frac{2-6}{\sqrt{4+9}}\right) \\ &= -\frac{4}{\sqrt{13}} = -\frac{4\sqrt{13}}{13} \end{aligned}$$

4. (c) From the Triangle Law for vector addition, we have $\mathbf{a}+\mathbf{c}=\mathbf{b}$, whence $\mathbf{c}=\mathbf{b}-\mathbf{a}$. Only statement (iii) is true.

5. (d) We have

$$\begin{aligned} & 3\mathbf{a}-2\mathbf{b} \\ = & 3\,[1,-4]-2\,[2,3] \\ = & [3,-12]-[4,6] \\ = & [-1,-18]\,. \end{aligned}$$

6. (b) Since $\sin^2 t+\cos^2 t=1$, we have

$$y=\sin^2 t = 1-(\cos t)^2 = 1-x^2.$$

So $y=1-x^2$. Thus the curve forms part of a parabola.

7. (d) The assertion that $\lim_{x\to 1} f(x)=2$ is false since the left-hand limit (1) differs from the right-hand limit (2).

8. (d) For the given hypotheses, it is always true that if $\lim_{x\to 3} f(x)=2$, then f is continuous at $x=3$.

9. (b) Since $\lim_{x\to 0} f(x)=3$ and $f(0)=1$, we conclude that f is not continuous at $x=0$ because $\lim_{x\to 0} f(x)$ does not equal $f(0)$.

10. (c) Now c is a solution of $g(c)=c^3+2c-12=\pi$ if it is a solution of $f(c)=g(c)-\pi=c^3+2c-12-\pi=0$. Note that $f(2)=-\pi<0$ and $f(3)=21-\pi>0$. Moreover, f (a polynomial) is continuous on $\mathbb{R}$. Therefore, by the Intermediate Value Theorem (IVT) there is a value of $c\in(2,3)\subset[2,3]$ such that $f(c)=0$ and thus $g(c)=\pi$.

11. (a) Recall that $\sqrt{x^2}=|x|=-x$ for $x<0$. Therefore,

$$\begin{aligned} & \lim_{x\to-\infty}\frac{\sqrt{x^2+4x}}{4x+1} \\ = & \lim_{x\to-\infty}\frac{|x|\sqrt{1+\frac{4}{x}}}{4x+1} \\ = & \lim_{x\to-\infty}\frac{-x\sqrt{1+\frac{4}{x}}}{4x+1} \\ = & \lim_{x\to-\infty}\frac{-\sqrt{1+\frac{4}{x}}}{4+\frac{1}{x}} = -\frac{1}{4}. \end{aligned}$$

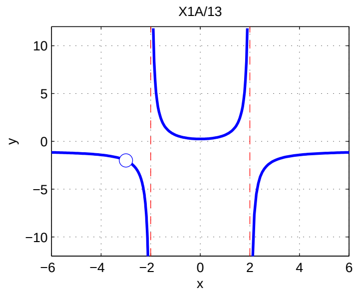
12. (a) Note that as $x\to 2^+$, the expression $3x-6$ is positive. Hence

$$\begin{aligned} & \lim_{x\to 2^+}\frac{|3x-6|}{6-3x} \\ = & \lim_{x\to 2^+}\frac{3x-6}{-(3x-6)} \\ = & \lim_{x\to 2^+}(-1)=-1. \end{aligned}$$

13. (b) Recall that $f(x)=\dfrac{\left(x^2+1\right)(x+3)}{\left(4-x^2\right)(x+3)}=\dfrac{x^2+1}{4-x^2}$ for $x\neq -3,-2,2$.

- As $x\to\pm\infty$, we see that $f(x)=\dfrac{1+\frac{1}{x^2}}{\frac{4}{x^2}-1}\to -1$. Therefore, $y=-1$ is a horizontal asymptote.
- As $x\to 2^-$, we see that $f(x)=\dfrac{x^2+1}{4-x^2}\to\dfrac{5}{0^+}=+\infty$. Hence $x=2$ is a vertical asymptote. [Also, as $x\to 2^+$, we see that $f(x)=\dfrac{x^2+1}{4-x^2}\to\dfrac{5}{0^-}=-\infty$. Therefore, (e) is false.]
- Similarly, $\lim_{x\to-2^-} f(x)=\dfrac{5}{0^+}=+\infty$. So $x=-2$ is a vertical asymptote.
- Finally, $f(x)=\dfrac{x^2+1}{4-x^2}\to\dfrac{10}{-5}=-2\neq\pm\infty$ as $x\to-3$. Therefore, $x=-3$ is not a vertical asymptote. Indeed, there is a removable discontinuity at $x=-3$. (Please see graph on reverse.)

- We conclude that (b) is the true statement. One look at the following graph tells the story!



14. (a) Rewrite as $y = 2 - 3x^{-1} + 4x^{-2}$. Then $y' = 3x^{-2} - 8x^{-3}$.
 (b) Rewrite as $f(x) = x + 2^{1/5}x^{2/5}$. Then $f'(x) = 1 + \frac{2}{5}\left(2^{1/5}\right)x^{-3/5}$.
 (c) The Quotient Rule gives
$$g'(s) = \frac{(6s+5)(4) - (4s-7)(6)}{(6s+5)^2} \text{ or } \frac{62}{(6s+5)^2}.$$
 (d) The Product Rule yields
$$\begin{aligned} h'(t) &= \left(3t^2 - 10t + 6\right)\left(t^4 + t^3 + t^2 + t\right) \\ &\quad + \left(t^3 - 5t^2 + 6t + 7\right)\left(4t^3 + 3t^2 + 2t + 1\right). \end{aligned}$$

15. (a) With $\mathbf{r}(t) = \left[4t^2, t^3 - 9t - 2\right]$, at time $t = 3$ the particle is at $(36, -2)$.
 (b) The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \left[8t, 3t^2 - 9\right]$. Thus $\mathbf{v}(3) = [24, 18]$ or $24\mathbf{i} + 18\mathbf{j}$.
 (c) Speed is the magnitude of velocity: $\|\mathbf{v}(3)\| = \sqrt{24^2 + 18^2}$ or 30 ft/s.
 (d) The particle's velocity is parallel to $\mathbf{i}$ when its $\mathbf{j}$-component is zero. Thus $3t^2 - 9 = 0$, which implies $t = \pm\sqrt{3}$ seconds.

16. (a) A direction vector for the line is
$$\mathbf{v} = \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [4, -5] - [2, 3] = [2, -8].$$
 Parametric equations are derived as follows.
$$\begin{aligned} \mathbf{L}(t) &= \overrightarrow{A} + t\mathbf{v} \\ [x(t), y(t)] &= [2, 3] + t[2, -8] \\ [x, y] &= [2t + 2, 3 - 8t] \end{aligned}$$
 (b) Write the line L in slope-intercept form: $y = \frac{2}{3}x - \frac{5}{3}$. Since slope = rise/run, a direction vector for the line is $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$, whose dot product with the vector $\mathbf{w} = 2\mathbf{i} - 3\mathbf{j}$ is zero. Hence $\mathbf{w}$ is perpendicular to $\mathbf{v}$ and hence to the line L. Thus statement (ii) is true.
 (c) Let $\mathbf{a} = [2, 5]$. The object of our desire is $\hat{\mathbf{a}}^{\perp}$, the orthogonal complement to the unit vector in the direction of $\mathbf{a}$. Now $\hat{\mathbf{a}} = \mathbf{a}/\|\mathbf{a}\| = [2, 5]/\sqrt{29} = \left[\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right]$, whence $\hat{\mathbf{a}}^{\perp} = \left[-\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right]$. (Note that $-\hat{\mathbf{a}}^{\perp}$ also fits the bill.)

17. (a) We have
$$\begin{aligned} \lim_{x\to 2} \frac{x^2 - 4}{x^2 + x - 6} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+3)} \\ &= \lim_{x\to 2} \frac{x+2}{x+3} = \frac{4}{5}. \end{aligned}$$
 (b) We have
$$\lim_{x\to\infty} \frac{1 + 2x - x^2}{1 - x + 2x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2} + \frac{2}{x} - 1}{\frac{1}{x^2} - \frac{1}{x} + 2} = -\frac{1}{2}.$$

18. The slope of our tangent line is the derivative of $f(x) = \dfrac{3x - 7}{x^2 + 5x - 4}$ evaluated at $x = 1$.
$$\begin{aligned} f'(x) &= \frac{\left(x^2 + 5x - 4\right)(3) - (3x - 7)(2x + 5)}{\left(x^2 + 5x - 4\right)^2} \\ f'(1) &= \frac{(2)(3) - (-4)(7)}{2^2} = \frac{6 + 28}{4} = \frac{34}{4} = \frac{17}{2} \end{aligned}$$
A point on the tangent line is $(1, f(1)) = (1, -2)$. Apply the point-slope formula and rearrange to the requested form.
$$\begin{aligned} y - (-2) &= \frac{17}{2}(x - 1) \\ 2y + 4 &= 17x - 17 \\ 17x - 2y &= 21 \text{ or equivalent} \end{aligned}$$

19. Use the definition of derivative and the limit laws.
$$\begin{aligned} \lim_{x\to 2} \frac{\left(f(x)\right)^2 - 16}{x - 2} &= \lim_{x\to 2} \frac{\left(f(x) - 4\right)\left(f(x) + 4\right)}{x - 2} \\ &= \lim_{x\to 2} \left(\frac{f(x) - f(2)}{x - 2} \cdot \left(f(x) + 4\right)\right) \\ &= f'(2)\left(f(2) + 4\right) \\ &= (4)(4 + 4) = 32 \end{aligned}$$

20. With $f(x) = \sqrt{x + 3}$, we have
$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \left(\frac{\sqrt{x+3} - 2}{x - 1} \cdot \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2}\right) \\ &= \lim_{x\to 1} \frac{(x+3) - 4}{(x-1)\left(\sqrt{x+3} + 2\right)} \\ &= \lim_{x\to 1} \frac{1}{\sqrt{x+3} + 2} = \frac{1}{4}. \end{aligned}$$