Fall 2006 Math 151
Exam 1B: Solutions
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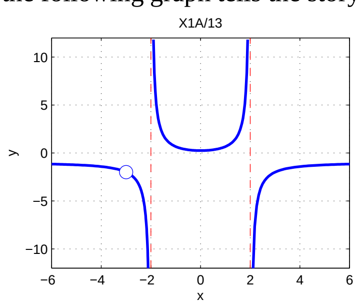
1. (d) Since $\sin^2 t + \cos^2 t = 1$, we have
$$y = \sin^2 t = 1 - (\cos t)^2 = 1 - x^2.$$
So $y = 1 - x^2$. Thus the curve forms part of a parabola.

2. (d) Recall that $f(x) = \dfrac{(x^2+1)(x+3)}{(4-x^2)(x+3)} = \dfrac{x^2+1}{4-x^2}$ for $x \neq -3, -2, 2$.
   - As $x \to \pm\infty$, we see that $f(x) = \dfrac{1 + \frac{1}{x^2}}{\frac{4}{x^2} - 1} \to -1$. Therefore, $y = -1$ is a horizontal asymptote.
   - As $x \to 2^-$, we see that $f(x) = \dfrac{x^2+1}{4-x^2} \to \dfrac{5}{0^+} = +\infty$. Hence $x = 2$ is a vertical asymptote. [Also, as $x \to 2^+$, we see that $f(x) = \dfrac{x^2+1}{4-x^2} \to \dfrac{5}{0^-} = -\infty$. Therefore, (e) is false.]
   - Similarly, $\displaystyle\lim_{x \to -2^-} f(x) = \frac{5}{0^+} = +\infty$. So $x = -2$ is a vertical asymptote.
   - Finally, $f(x) = \dfrac{x^2+1}{4-x^2} \to \dfrac{10}{-5} = -2 \neq \pm\infty$ as $x \to -3$. Therefore, $x = -3$ is not a vertical asymptote. Indeed, there is a removable discontinuity at $x = -3$.
   - We conclude that (d) is the true statement. One look at the following graph tells the story!



3. (a) Recall that $\sqrt{x^2} = |x| = -x$ for $x < 0$. Therefore,
$$\begin{aligned}
&\lim_{x \to -\infty} \frac{\sqrt{x^2+4x}}{4x+1} \\
=\ &\lim_{x \to -\infty} \frac{|x|\sqrt{1+\frac{4}{x}}}{4x+1} \\
=\ &\lim_{x \to -\infty} \frac{-x\sqrt{1+\frac{4}{x}}}{4x+1} \\
=\ &\lim_{x \to -\infty} \frac{-\sqrt{1+\frac{4}{x}}}{4+\frac{1}{x}} = -\frac{1}{4}.
\end{aligned}$$

4. (e) Now c is a solution of $g(c) = c^3 + 2c - 12 = \pi$ if it is a solution of $f(c) = g(c) - \pi = c^3 + 2c - 12 - \pi = 0$. Note that $f(2) = -\pi < 0$ and $f(3) = 21 - \pi > 0$. Moreover, f (a polynomial) is continuous on $\mathbb{R}$. Therefore, by the Intermediate Value Theorem (IVT) there is a value of $c \in (2,3) \subset [2,3]$ such that $f(c) = 0$ and thus $g(c) = \pi$.

5. (b) The assertion that $\displaystyle\lim_{x \to 1} f(x) = 2$ is false since the left-hand limit (1) differs from the right-hand limit (2).

6. (b) From the Triangle Law for vector addition, we have $\mathbf{a} + \mathbf{c} = \mathbf{b}$, whence $\mathbf{c} = \mathbf{b} - \mathbf{a}$. Only statement (iii) is true.

7. (d) We have
$$\begin{aligned}
&3\mathbf{a} - 2\mathbf{b} \\
=\ &3[1,-4] - 2[2,3] \\
=\ &[3,-12] - [4,6] \\
=\ &[-1,-18].
\end{aligned}$$

8. (e) We have
$$\begin{aligned}
&\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \\
=\ &\lim_{h \to 0} \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)} \\
=\ &\lim_{h \to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6}.
\end{aligned}$$

9. (d) Use the definition and properties of the dot product.
$$\begin{aligned}
&\mathbf{v} \cdot (2\mathbf{w} - 3\mathbf{v}) \\
=\ &2\mathbf{v} \cdot \mathbf{w} - 3\mathbf{v} \cdot \mathbf{v} \\
=\ &2|\mathbf{v}||\mathbf{w}|\cos 60^\circ - 3|\mathbf{v}|^2 \\
=\ &2(2)(3)\left(\tfrac{1}{2}\right) - 3(2)^2 \\
=\ &6 - 12 = -6
\end{aligned}$$

10. (b) For the given hypotheses, it is always true that if $\displaystyle\lim_{x \to 3} f(x) = 2$, then f is continuous at $x = 3$.

11. (a) With $\mathbf{a} = [2,-3]$ and $\mathbf{b} = [1,2]$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is
$$\begin{aligned}
\text{comp}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}\right) \\
&= \left(\frac{2-6}{\sqrt{4+9}}\right) \\
&= -\frac{4}{\sqrt{13}} = -\frac{4\sqrt{13}}{13}
\end{aligned}$$

12. (e) Note that as $x \to 2^+$, the expression $3x - 6$ is positive. Hence
$$\begin{aligned}
&\lim_{x \to 2^+} \frac{|3x-6|}{6-3x} \\
=\ &\lim_{x \to 2^+} \frac{3x-6}{-(3x-6)} \\
=\ &\lim_{x \to 2^+} (-1) = -1.
\end{aligned}$$

13. (d) Since $\lim_{x\to 0} f(x) = 3$ and $f(0) = 1$, we conclude that f is not continuous at $x = 0$ because $\lim_{x\to 0} f(x)$ does not equal $f(0)$.

14. (a) Rewrite as $y = 4 - 2x^{-1} + 3x^{-2}$. Then $y' = 2x^{-2} - 6x^{-3}$.

(b) Rewrite as $f(x) = 3x + 2^{1/4}x^{5/4}$. Then $f'(x) = 3 + \frac{5}{4}\left(2^{1/4}\right)x^{1/4}$.

(c) The Quotient Rule gives

$$g'(s) = \frac{(8s+2)(7) - (7s-3)(8)}{(8s+2)^2} \text{ or } \frac{38}{(8s+2)^2}.$$

(d) The Product Rule yields

$$\begin{aligned} h'(t) &= \left(3t^2 - 10t^4 - 7\right)\left(t^7 + t^5 + t^3 + t\right) \\ &\quad + \left(t^3 - 2t^5 - 7t + 6\right)\left(7t^6 + 5t^4 + 3t^2 + 1\right). \end{aligned}$$

15. (a) With $\mathbf{r}(t) = \left[4t^2, t^3 - 9t - 2\right]$, at time $t = 2$ the particle is at $(16, -12)$.

(b) The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \left[8t, 3t^2 - 9\right]$. Thus $\mathbf{v}(3) = [16, 3]$ or $16\mathbf{i} + 3\mathbf{j}$.

(c) Speed is the magnitude of velocity: $\|\mathbf{v}(3)\| = \sqrt{16^2 + 3^2}$ or $\sqrt{265}$ ft/s.

(d) The particle's velocity is parallel to $\mathbf{j}$ when its $\mathbf{i}$-component is zero. Thus $8t = 0$, which implies $t = 0$ seconds.

16. (a) A direction vector for the line is

$$\mathbf{v} = \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [-6, 2] - [4, 9] = [-10, -7].$$

Parametric equations are derived as follows.

$$\begin{aligned} \mathbf{L}(t) &= \overrightarrow{A} + t\mathbf{v} \\ [x(t), y(t)] &= [4, 9] + t[-10, -7] \\ [x, y] &= [4 - 10t, 9 - 75] \end{aligned}$$

(b) Write the line L in slope-intercept form: $y = -\frac{3}{2}x + \frac{5}{2}$. Since slope = rise/run, a direction vector for the line is $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$. Hence $\mathbf{v}$ is parallel to the line L. Thus statement (i) is true.

(c) Let $\mathbf{a} = [-2, 5]$. The object of our desire is $\hat{\mathbf{a}}^{\perp}$, the orthogonal complement to the unit vector in the direction of $\mathbf{a}$. Now $\hat{\mathbf{a}} = \mathbf{a}/\|\mathbf{a}\| = [-2, 5]/\sqrt{29} = \left[\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right]$, whence $\hat{\mathbf{a}}^{\perp} = \left[-\frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\right]$. (Note that $-\hat{\mathbf{a}}^{\perp}$ also fits the bill.)

17. (a) We have

$$\begin{aligned} \lim_{x\to -2} \frac{x^2 - 4}{x^2 + x - 6} &= \lim_{x\to -2} \frac{(x-2)(x+2)}{(x+2)(x-3)} \\ &= \lim_{x\to -2} \frac{x-2}{x-3} = \frac{-4}{-5} = \frac{4}{5}. \end{aligned}$$

(b) We have

$$\lim_{x\to\infty} \frac{1 + 2x + 3x^2}{1 - x - x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2} + \frac{2}{x} + 3}{\frac{1}{x^2} - \frac{1}{x} - 1} = -3.$$

18. The slope of our tangent line is the derivative of $f(x) = \dfrac{2x - 7}{x^2 + 6x - 4}$ evaluated at $x = 1$.

$$\begin{aligned} f'(x) &= \frac{\left(x^2 + 6x - 4\right)(2) - (2x - 7)(2x + 6)}{\left(x^2 + 6x - 4\right)^2} \\ f'(1) &= \frac{(3)(2) - (-5)(8)}{3^2} = \frac{46}{9} \end{aligned}$$

A point on the tangent line is $(1, f(1)) = \left(1, -\frac{5}{3}\right)$. Apply the point-slope formula and rearrange to the requested form.

$$\begin{aligned} y - \left(-\frac{5}{3}\right) &= \frac{46}{9}(x - 1) \\ 9y + 15 &= 46x - 46 \\ 46x - 9y &= 61 \text{ or equivalent} \end{aligned}$$

19. Use the definition of derivative and the limit laws.

$$\begin{aligned} \lim_{x\to 3} \frac{\left(f(x)\right)^2 - 25}{x - 3} &= \lim_{x\to 3} \frac{\left(f(x) - 5\right)\left(f(x) + 5\right)}{x - 3} \\ &= \lim_{x\to 3} \left(\frac{f(x) - f(3)}{x - 3} \cdot \left(f(x) + 5\right)\right) \\ &= f'(3)\left(f(3) + 5\right) \\ &= (5)(5 + 5) = 50 \end{aligned}$$

20. With $f(x) = \sqrt{x + 8}$, we have

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \left(\frac{\sqrt{x+8} - 3}{x - 1} \cdot \frac{\sqrt{x+8} + 3}{\sqrt{x+8} + 3}\right) \\ &= \lim_{x\to 1} \frac{(x + 8) - 9}{(x - 1)\left(\sqrt{x+8} + 3\right)} \\ &= \lim_{x\to 1} \frac{1}{\sqrt{x+8} + 3} = \frac{1}{6}. \end{aligned}$$