Fall 2006 Math 151
Exam 3A: Solutions
Fri, 01/Dec ©2006, Art Belmonte

1. (b) We have

$$\lim_{x\to 2}\frac{x^3-12x+16}{x^3-3x^2+4} \overset{\text{L'H}}{=} \lim_{x\to 2}\frac{3x^2-12}{3x^2-6x} \overset{\text{L'H}}{=} \lim_{x\to 2}\frac{6x}{6x-6}=\frac{12}{6}=2.$$

2. (c) Recall that $y=e^{-x^2/2}$. Find where $y''<0$.

$$\begin{aligned} y' &= e^{-x^2/2}(-x) = -xe^{-x^2/2} \\ y'' &= -\left(e^{-x^2/2}+xe^{-x^2/2}(-x)\right) \\ y'' &= \left(x^2-1\right)e^{-x^2/2} \end{aligned}$$

Since $e^{-x^2/2}>0$ for $x\in\mathbb{R}$, we see that $y''<0$ where $x^2-1<0$; i.e., $x^2<1$ or $-1<x<1$.

3. (c) Let $\theta=\tan^{-1}2$. Draw a right triangle. Via trigonometry, we see that $\cos\theta=1/\sqrt{5}$. (See picture on reverse.)

4. (d) The sum $\displaystyle\sum_{n=3}^{2006}\frac{1}{\sqrt{n}}$ fits the bill.

5. (a) The integral in question represents the area under f and above the x-axis. Just add areas of geometrical shapes.

$$(3\times 5)+(3\times 1)+\tfrac{1}{2}(3\times 4)=15+3+6=24$$

6. (c) We have $\dfrac{d}{dx}\left(\tan^{-1}(\sin x)\right)=\dfrac{1}{1+(\sin x)^2}\cdot\cos x$.

7. (b) Since $\frac{d}{dx}\left(\frac{1}{4}x^4+2x^3+9\right)=x^3+6x^2$, we conclude that $F(x)=\frac{1}{4}x^4+2x^3+9$ is an antiderivative of the function $f(x)=x^3+6x^2$.

8. (e) Let y be the temperature of the turkey at time t. Using the law of exponential growth/decay, we have that $dy/dt=-\frac{1}{2}y$ implies $y=y_0e^{-t/2}=57e^{-t/2}$. Thus $y(\ln 9)=57e^{-\frac{1}{2}\ln 9}=57e^{\ln\frac{1}{3}}=57\left(\frac{1}{3}\right)=19\ {}^\circ\text{C}$.

9. (a) Use logarithmic differentiation.

$$\begin{aligned} y &= (1+x)^x \\ \ln y &= x\ln(1+x) \\ \frac{1}{y}\frac{dy}{dx} &= (1)\ln(1+x)+x\left(\frac{1}{1+x}\right) \\ \frac{dy}{dx} &= (1+x)^x\left(\ln(1+x)+\frac{x}{1+x}\right) \\ \frac{dy}{dx} &= (1+x)^x\ln(1+x)+x(1+x)^{x-1} \end{aligned}$$

10. (a) Recall $f(x)=\left(x^2-6x+10\right)^{-1}$. Thus $f'(x)=-\dfrac{2x-6}{\left(x^2-6x+10\right)^2}$ for all $x\in\mathbb{R}$ since $x^2-6x+10=x^2-6x+9+1=(x-3)^2+1>0$. Hence $f'(x)=0$ implies $2x-6=0$; i.e., $x=3\in[0,4]$ is the lone critical number of f. Crank out function values at critical numbers and endpoints: $f(0)=1/10$, $f(3)=1$, $f(4)=1/2$. We conclude that the absolute maximum value of f on $[0,4]$ is $f(3)=1$.

11. (e) Curve 1 is f''; curve 2 is f; curve 3 is f'.

12. (a) Using properties of summation, we have

$$\sum(5f+2g)=5\sum f+2\sum g=5(101)+2(35)=575.$$

13. (c) This is an indeterminate power, 1^∞.

$$\begin{aligned} y &= (\cos x)^{1/x^2} \\ \ln y &= \frac{\ln(\cos x)}{x^2} \\ \lim_{x\to 0}\ln y &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x}=\lim_{x\to 0}\frac{-\tan x}{2x} \\ \lim_{x\to 0}\ln y &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{-\sec^2 x}{2}=-\frac{1}{2} \end{aligned}$$

Therefore, $\lim y=\lim e^{\ln y}=e^{-1/2}$.

14. Change this indeterminate difference into an indeterminate quotient, then apply l'Hospital's Rule—twice.

$$\begin{aligned} \lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{1-(1+x)^{-1}}{(1)\ln(1+x)+\frac{x}{1+x}} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{(1+x)^{-2}}{\frac{1}{1+x}+\frac{(1+x)(1)-x(1)}{(1+x)^2}} \\ &= \frac{1}{1+1}=\frac{1}{2} \end{aligned}$$

15. (a) Now $y=9\ln(1+x)+\frac{1}{2}\ln x-\frac{1}{7}\ln(1+3x)$, whence $y'=\dfrac{9}{1+x}+\dfrac{1}{2x}-\dfrac{3}{7(1+3x)}$.

(b) With $f(x)=e^{2x}+(2x)^e+e^\pi+(2+e)^x$, we have $f'(x)=2e^{2x}+e(2x)^{e-1}(2)+(2+e)^x\ln(2+e)$.

(c) Since $g(x)=\ln|x|+\ln\left(\ln\left(x^2\right)\right)+\log_{10}\left(x^2+2\right)$, we have $g'(x)=\dfrac{1}{x}+\dfrac{1}{\ln\left(x^2\right)}\cdot\dfrac{2}{x}+\dfrac{2x}{\left(x^2+2\right)\ln 10}$.

16. The Mean Value Theorem states that if f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a value $c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$. [In other words, the slope of the tangent line to f at c equals the slope of the secant line through $(a,f(a))$ and $(b,f(b))$.]

17. • First determine the velocity function.

$$\begin{aligned}
\mathbf{v}'(t)=\mathbf{a}(t) &= [t,1]\\
\mathbf{v}(t) &= \left[\tfrac{1}{2}t^2, t\right]+\mathbf{C}\\
[1,-1]=\mathbf{v}(0) &= [0,0]+\mathbf{C}\\
\mathbf{C} &= [1,-1]-[0,0]=[1,-1]\\
\mathbf{v}(t) &= \left[\tfrac{1}{2}t^2+1, t-1\right]
\end{aligned}$$

• Then determine the position function.

$$\begin{aligned}
\mathbf{r}'(t)=\mathbf{v}(t) &= \left[\tfrac{1}{2}t^2+1, t-1\right]\\
\mathbf{r}(t) &= \left[\tfrac{1}{6}t^3+t, \tfrac{1}{2}t^2-t\right]+\mathbf{K}\\
[1,2]=\mathbf{r}(1) &= \left[\tfrac{7}{6}, -\tfrac{1}{2}\right]+\mathbf{K}\\
\mathbf{K} &= [1,2]-\left[\tfrac{7}{6}, -\tfrac{1}{2}\right]=\left[-\tfrac{1}{6}, \tfrac{5}{2}\right]\\
\mathbf{r}(t) &= \left[\tfrac{1}{6}t^3+t-\tfrac{1}{6}, \tfrac{1}{2}t^2-t+\tfrac{5}{2}\right]
\end{aligned}$$

18. Recall that $y=f(x)=\sin^{-1}x$.

(a) The domain of f is $[-1,1]$ or $-1\le x\le 1$.

(b) The range of f is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ or $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.

(c) We have $\sin^{-1}\left(\sin\frac{3}{2}\pi\right)=\sin^{-1}(-1)=-\frac{\pi}{2}$.

(d) Since $f'(x)=1/\sqrt{1-x^2}$, we see that $f'(0)=1$.

(e) Via trig, $\cos\left(\sin^{-1}\frac{1}{4}\right)=\sqrt{15}/4$. (Picture at bottom.)

(f) Finally, $\sin\left(\sin^{-1}\frac{1}{4}\right)=\frac{1}{4}$.

19. Recall that $f(x)=x^3-3x^2-24x-22$.

(a) The critical numbers of f occur where the derivative of f is zero or is undefined. Since $f'(x)=3x^2-6x-24=3\left(x^2-2x-8\right)=3(x+2)(x-4)$ is defined for all x, the critical numbers of f are $x=-2,4$, where the derivative is zero.

(b)
- The point $(-2, f(-2))$ is a local maximum of f by the First Derivative Test since the sign of f' changes from $+$ to $-$ as x increases through -2. (Alternatively, use the Second Derivative Test. Now $f''(x)=6x-6$, so $f''(-2)=-18<0$. This signifies a local maximum.)
- The point $(4, f(4))$ is a local minimum of f by the First Derivative Test since the sign of f' changes from $-$ to $+$ as x increases through 4. (Alternatively, use the Second Derivative Test. Now $f''(4)=18>0$, which signifies a local minimum.)

(c) An inflection point of f occurs where the second derivative changes sign. Now $f''(x)=6x-6=0$ implies $x=1$. Note that f'' changes sign from $-$ to $+$ as x increases through 1. Therefore, $(1, f(1))$ is an inflection point of f. [Graph of f at right.→]

(d) The Extreme Value Theorem guarantees that the continuous function f attains an absolute maximum on the closed interval $[-1,0]$. From part (a), we see that there are no critical numbers of f in this interval. Accordingly, the absolute maximum must occur at an endpoint of the interval. Now $f(-1)=-2$, whereas $f(0)=-22$. Hence the absolute maximum of f on $[-1,0]$ is -2, which occurs at $x=-1$.

20. Place the house at the origin of the xy-plane. Let t be the time in hours after the couple has started walking. The husband's position is $[0,y]=[0,4t-10]$, whereas that of the wife is $[x,0]=[3t,0]$. By the Pythagorean Theorem, the distance between them is $z=\sqrt{x^2+y^2}=\left(x^2+y^2\right)^{1/2}$. [NOTE: This distance z is always positive since otherwise $x=y=0$, whence $3t=0$ and $4t-10=0$, from which $0=t=\frac{5}{2}$, a contradiction.]

(a) To find the time they are closest together, solve $dz/dt=0$, and verify that this gives the absolute minimum distance.

$$\begin{aligned}
\frac{dz}{dt} &= 0\\
\frac{1}{2}\left(x^2+y^2\right)^{-1/2}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right) &= 0\\
\frac{x\frac{dx}{dt}+y\frac{dy}{dt}}{\sqrt{x^2+y^2}} &= 0\\
\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)/z &= 0\\
\big((3t)(3)+(4t-10)(4)\big)/z &= 0\\
dz/dt=(25t-40)/z(t) &= 0\\
t &= \tfrac{8}{5}\text{ hr}
\end{aligned}$$

Note that $dz/dt<0$ for $t<\frac{8}{5}$,whereas $dz/dt>0$ for $t>\frac{8}{5}$. The First Derivative Test for Absolute Extrema allows us to conclude that $\left(\frac{8}{5}, z\left(\frac{8}{5}\right)\right)$ is the absolute minimum; i.e., the couple are closest together when $t=\frac{8}{5}$ hr or 96 minutes (1 hour, 36 minutes).

(b) Recall that the distance between the couple at time t is

$$z=\sqrt{x^2+y^2}=\sqrt{(3t)^2+(4t-10)^2}=\sqrt{25t^2-80t+100}.$$

At time $t=\frac{8}{5}$, we have

$$z=\sqrt{64-128+100}=\sqrt{36}=6\text{ miles}.$$

This is the closest distance between the couple.