Fall 2007 Math 151
Exam 1 Supplement
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An alternative way to do problems X1A/14(b) and X1B/15(b) is to use advanced theory from later in the course. Many people tried to do this; most left out important pieces. Here is the gist. Recall the piecewise definitition of f.

$$f(x) = \begin{cases} 2\,|x-2| & \text{if } x < 2, \\ (x-3)^2 - 1 & \text{if } 2 \le x \le 4, \\ 2x - 4 & \text{if } x > 4 \end{cases}$$

Note that for $x < 2$ we have $f(x) = -2(x-2) = 4 - 2x$, while for $2 < x < 4$ we have $f(x) = x^2 - 6x + 8$. Thus f is continuous on [say] $(1, 2) \cup (2, 4)$ since polynomials are continuous.

- Now $\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} (-2) = -2$.
- Moreover, $\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+} (2x - 6) = -2$.
- Thus $\lim_{x\to 2} f'(x) = -2$.
- Since $\lim_{x\to 2} f(x) = 0 = f(2)$, we see that f is continuous at $x = 2$. Therefore, f is continuous on [say] $(1, 4)$, an open interval containing $x = 2$.
- With these four pieces, the Assertion (see below) allows us to conclude that $f'(2) = -2$.

People made tacit use of the Assertion. While it is true and even geometrically plausible, it is not entirely obvious. (Without the continuity hypothesis it is *not* true.) **So please read the Assertion and its proof carefully. (Or dispense with it altogether and do the problem directly as outlined in original exam solutions!)**

Assertion

Let f be continuous on (a, b) with [finite] derivative everywhere in (a, b), except possibly at p. If $\lim_{x\to p} f'(x)$ exists and is equal to A, then $f'(p)$ must also exist and be equal to A.

Proof

- First observe that $a < p < b$ since p is in the open interval (a, b).
- Let $u \in (a, p)$; i.e., u is in the interval (a, p). Then by hypothesis f is continuous on $[u, p]$ and differentiable on (u, p). By the Mean Value Theorem [MVT], there is a number $w \in (u, p)$ such that

$$f'(w) = \frac{f(p) - f(u)}{p - u} = \frac{f(u) - f(p)}{u - p}.$$

Hence as $u \to p^-$, we have $w \to p^-$ and thus

$$\frac{f(u) - f(p)}{u - p} = f'(w) \to A$$

since $\lim_{x\to p} f'(x) = A$ by hypothesis. In other words, $\displaystyle\lim_{u\to p^-} \frac{f(u) - f(p)}{u - p} = A$.

- Now repeat this argument for $v \in (p, b)$. Again by hypothesis, f is continuous on $[p, v]$ and differentiable on (p, v). By the MVT, there is a number $z \in (p, v)$ such that

$$f'(z) = \frac{f(v) - f(p)}{v - p}.$$

Hence as $v \to p^+$, we have $z \to p^+$ and thus

$$\frac{f(v) - f(p)}{v - p} = f'(z) \to A$$

since $\lim_{x\to p} f'(x) = A$ by hypothesis. In other words, $\displaystyle\lim_{v\to p^+} \frac{f(v) - f(p)}{v - p} = A$.

- Therefore, $f'(p) = \displaystyle\lim_{x\to p} \frac{f(x) - f(p)}{x - p} = A$ since the corresponding one-sided limits exist and are both equal to A.