Fall 2007 Math 151
Exam 1B: Solutions
Mon, 01/Oct ©2007 Art Belmonte

1. (e) Since $x = 2\sin t$ and $y = 4 + \cos t$, we have

$$1 = \sin^2 t + \cos^2 t = \left(\frac{x}{2}\right)^2 + (y-4)^2$$

or $\dfrac{(x-0)^2}{2^2} + \dfrac{(y-4)}{1^2} = 1$, an ellipse centered at $(0, 4)$, traversed clockwise.

2. (d) The limit is

$$\begin{aligned} & \lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3+1} \\ = & \lim_{x\to-\infty} \frac{|x|^3\sqrt{9-\frac{1}{x^5}}}{x^3+1} \\ = & \lim_{x\to-\infty} \frac{-x^3\sqrt{9-\frac{1}{x^5}}}{x^3+1} \\ = & \lim_{x\to-\infty} \frac{-\sqrt{9-\frac{1}{x^5}}}{1+\frac{1}{x^3}} = -3. \end{aligned}$$

3. (c) Let's equivalently find the interval in which the continuous function $f(x) = x + \cos x - 3$ is zero. Now $f(\pi) = \pi - 4 < 0$, whereas $f(2\pi) = 2\pi - 2 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for some c in $(\pi, 2\pi)$; i.e., $c + \cos c = 3$ for some c in $(\pi, 2\pi)$.

4. (d) With $\mathbf{a} = [5, -12]$ and $\mathbf{b} = [-3 - 6]$, we have $\mathbf{a} - \mathbf{b} = [8, -6]$, whence $\|\mathbf{a} - \mathbf{b}\| = \sqrt{64 + 36} = 10$.

5. (d) The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{x=2} = \left.2x\right|_{x=2} = 4$. So $\mathbf{v} = [1, 4]$ is a tangent vector to the curve at $(2, 4)$. Therefore, a unit tangent vector is $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\| = \left[\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right]$ or $\frac{1}{\sqrt{17}}\,\mathbf{i} + \frac{4}{\sqrt{17}}\,\mathbf{j}$.

6. (a) We have

$$\begin{aligned} \lim_{x\to16} \frac{4-\sqrt{x}}{16x - x^2} &= \lim_{x\to16} \frac{\left(4-\sqrt{x}\right)}{x\left(4-\sqrt{x}\right)\left(4+\sqrt{x}\right)} \\ &= \lim_{x\to16} \frac{1}{x\left(4+\sqrt{x}\right)} \\ &\to \lim_{x\to16} \frac{1}{16(4+4)} = \frac{1}{128}. \end{aligned}$$

7. (e) The average velocity is

$$\begin{aligned} v_{avg} &= \frac{y(1.1) - y(1.0)}{1.1 - 1.0} \\ &= \frac{(11 - 2.42) - (10 - 2)}{0.1} \\ &= 5.80 \text{ m/s}. \end{aligned}$$

8. (e) The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\begin{aligned} \text{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= \left(\frac{-4+2}{\sqrt{5}}\right)\frac{[1, 2]}{\sqrt{5}} \\ &= \left[-\frac{2}{5}, -\frac{4}{5}\right]. \end{aligned}$$

9. (e) Now $y = \dfrac{x^2 - 2x}{x^2 - x - 2} = \dfrac{x(x-2)}{(x+1)(x-2)}$, so candidates for vertical asymptotes are $x = -1$ and $x = 2$.

- Observe that $\lim_{x\to-1^+} y = \lim_{x\to-1^+} \dfrac{x}{x+1} = -\infty$. Hence $x = -1$ is a vertical asymptote.
- However, since $\lim_{x\to2} y = \lim_{x\to2} \dfrac{x}{x+1} = \dfrac{2}{3} \neq \pm\infty$, we conclude that $x = 2$ is not a vertical asymptote.

10. (c) The work done is

$$\begin{aligned} W &= \mathbf{F}\cdot\mathbf{D} \\ &= \|\mathbf{F}\|\,\|\mathbf{D}\|\cos\theta \\ &= (1500)(1000)\cos 30^\circ \\ &= 750,000\sqrt{3} \text{ joules}. \end{aligned}$$

11. (e) As $x \to 0.5^-$, we have

$$\frac{2x-1}{\left|2x^3 - x^2\right|} = \frac{(2x-1)}{-x^2(2x-1)} = \frac{-1}{x^2} \to -4.$$

12. (b) Now $f(x) = 4x^2 - x^3$ implies $f'(x) = 8x - 3x^2$. Thus $f(3) = 36 - 27 = 9$ and $f'(3) = 24 - 27 = -3$. The point-slope formula then yields

$$\begin{aligned} y - 9 &= -3(x-3) \\ y &= -3x + 18. \end{aligned}$$

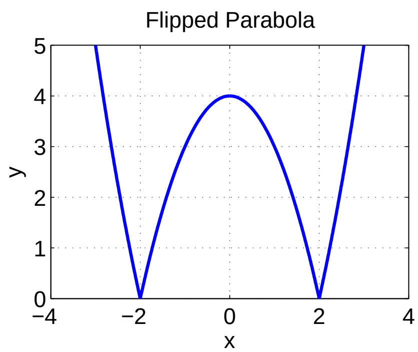
13. Rewrite $g(x) = \left|x^2 - 4\right|$ as a piecewise-defined function.

$$g(x) = \begin{cases} 4 - x^2 & \text{if } |x| < 2, \\ 0 & \text{if } |x| = 2, \\ x^2 - 4 & \text{if } |x| > 2 \end{cases}$$

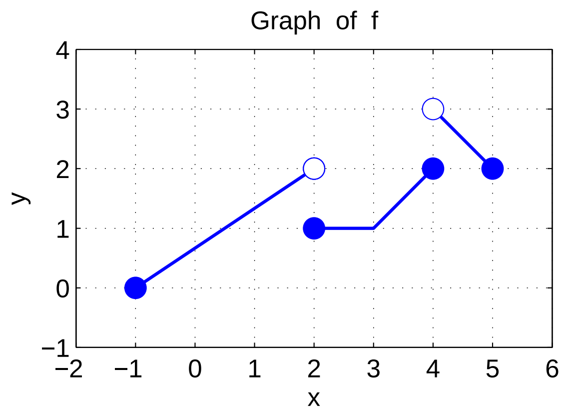
- Since g is a piecewise polynomial for $|x| \neq 2$, we have

$$g'(x) = \begin{cases} -2x & \text{if } |x| < 2, \\ 2x & \text{if } |x| > 2. \end{cases}$$

- One glance at the graph of g will convince you that g is not differentiable at $x = \pm 2$ since the graph is not smooth there. Therefore the domain of g is $|x| \neq 2$; i.e., $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.



14. Recall that $g(x) = x^2$ and that f was graphically depicted as shown below.



(a) From the graph of f, we have $\lim_{x\to 2^-} f(x) = 2$.

(b) We have $\lim_{x\to 2^+} g(f(x)) = \lim_{x\to 2^+} (f(x))^2 = 1^2 = 1$.

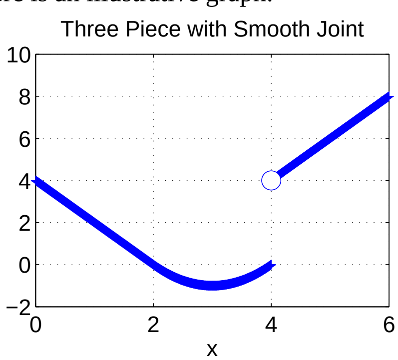
(c) We have

$$\lim_{x\to 2^+} f(g(x)) = \lim_{x\to 2^+} f\left(x^2\right) = \lim_{w\to 4^+} f(w) = 3.$$

15. Recall the piecewise definitition of f.

$$f(x) = \begin{cases} 2|x-2| & \text{if } x < 2, \\ (x-3)^2 - 1 & \text{if } 2 \le x \le 4, \\ 2x-4 & \text{if } x > 4 \end{cases}$$

Here is an illustrative graph.



(a)
- For $E = (-\infty, 2) \cup (2, 4) \cup (4, \infty)$, f is the composition of continuous functions and hence is continuous on E.
- As $x \to 2^-$, we have $f(x) = 2|x-2| \to 0$. Moreover, as $x \to 2^+$, we have $f(x) = (x-3)^2 - 1 \to 0$. Hence $\lim_{x\to 2} f(x) = 0 = f(2)$. Thus f is continuous at $x = 2$.
- As $x \to 4^+$, we have $f(x) = 2x - 4 \to 4 \neq 0 = f(4)$. Therefore, f is discontinuous at $x = 4$.

(b)
- As $x \to 2^-$. we have

$$\frac{f(x) - f(2)}{x-2} = \frac{-2(x-2) - 0}{x-2} = -2 \to -2.$$

- Moreover, as $x \to 2^+$, we have

$$\frac{f(x) - f(2)}{x-2} = \frac{\left((x-3)^2 - 1\right) - 0}{x-2} = \frac{x^2 - 6x + 8}{x-2} = \frac{(x-2)(x-4)}{(x-2)} = x - 4 \to -2.$$

In other words,

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = -2$$

and thus f is differentiable at $x = 2$.

16. Let $f(x) = \sqrt{1+2x}$. We have

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \\ &= \lim_{x\to a} \frac{\sqrt{1+2x} - \sqrt{1+2a}}{x-a} \\ &= \lim_{x\to a} \frac{(1+2x) - (1+2a)}{(x-a)\left(\sqrt{1+2x} + \sqrt{1+2a}\right)} \\ &= \lim_{x\to a} \frac{2(x-a)}{(x-a)\left(\sqrt{1+2x} + \sqrt{1+2a}\right)} \\ &= \lim_{x\to a} \frac{2}{\left(\sqrt{1+2x} + \sqrt{1+2a}\right)} \\ &= \frac{2}{2\sqrt{1+2a}} = \frac{1}{\sqrt{1+2a}}. \end{aligned}$$

17. Derivative rules yield

(a) $h'(t) = \left(6t^2 - 3t^{-1/4} + 8\right)\left(t^4 + t^{1/3} + 45\right) + \left(2t^3 - 4t^{3/4} + 8t - 7\right)\left(4t^3 + \frac{1}{3}t^{-2/3}\right)$ and

(b) $q'(x) = \dfrac{\left(8x^3 + 4x - 2\right)(5) - (5x+1)\left(24x^2 + 4\right)}{\left(8x^3 + 4x - 2\right)^2}$.

Note

An alternative way to do Problem 15(b) is to use advanced theory from later in the course. Many people attempted to do this; most of them left out important pieces. For full details, please read the **X1 Supplement** carefully. (Or just do the problem directly as outlined in the solutions and be done with it!)