Math 151 Fall 2008 Exam II
Solutions-Form B

1. C: Let $y = \dfrac{1-x}{4x+3}$. Interchange x and y:
$x = \dfrac{1-y}{4y+3}$. Solve for y: $x(4y+3) = 1-y$
$4xy + y = 1 - 3x$, hence $y = \dfrac{1-3x}{4x+1}$.

2. E: $\lim_{x\to 0^-} e^{1/x} = 0$ since $\lim_{x\to 0^-} 1/x = -\infty$
and $\lim_{x\to 0^-} e^{1/x} = e^{\left(\lim_{x\to 0^-} 1/x\right)} = \lim_{y\to -\infty} e^y = 0$

3. B: The quadratic approximation for $f(x)$ at $x = 1$ is $Q(x) = f(1) + f'(1)(x-1) + \dfrac{f''(1)}{2}(x-1)^2$.
$f(1) = 2$, $f'(1) = -2$ and $f''(1) = 4$. Thus
$Q(x) = 2 - 2(x-1) + 2(x-1)^2$, and
$Q(\frac{1}{2}) = 2 - 2(\frac{1}{2} - 1) + 2(\frac{1}{2} - 1)^2 = \dfrac{7}{2}$.

4. D: $f(t) = 2\sin t - 3\cos t$,
$v(t) = f'(t) = 2\cos t + 3\sin t$ and
$a(t) = v'(t) = -2\sin t + 3\cos t$. Evaluating these at $t = \dfrac{\pi}{6}$ yields $v\left(\dfrac{\pi}{6}\right) = \sqrt{3} + \dfrac{3}{2}$ and
$a\left(\dfrac{\pi}{6}\right) = -1 + \dfrac{3\sqrt{3}}{2}$

5. E: Let $f(x) = x^5 - 2x + 5$. Newton's Method says if x_1 is the first guess to $f(x) = 0$, then the second approximation is given by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. Thus if $x_1 = 1$, $x_2 = 1 - \dfrac{f(1)}{f'(1)} = 1 - 4/3 = -\dfrac{1}{3}$.

6. C: Recall if g is the inverse of f, then
$g'(a) = \dfrac{1}{f'(g(a))}$. Thus $g'(2) = \dfrac{1}{f'(g(2))}$.
Since $f(3) = 2$, $g(2) = 3$. $g'(2) = \dfrac{1}{f'(3)} = \dfrac{1}{7}$

7. A: To solve $\ln(x+e) + \ln(x-e) = 2 + \ln(3)$, we will use logarithm properities:
$\ln(x+e) + \ln(x-e) - \ln(3) = 2$
$\rightarrow \ln\left(\dfrac{(x+e)(x-e)}{3}\right) = 2$
$\rightarrow \dfrac{(x+e)(x-e)}{3} = e^2$
$\rightarrow (x+e)(x-e) = 3e^2$
$\rightarrow x^2 - e^2 = 3e^2$, yielding $x = \pm 2e$. Now, the domain of $\ln(x+e) + \ln(x-e) = 2 + \ln(3)$ is $x > e$, hence $x = -2e$ is extraneous, so the only solution is $x = 2e$.

8. A: First we will find the tangent vector at $t = 0$ and then make it a unit vector by dividing by the magnitude:
$\mathbf{r}(t) = \langle e^{2t}, t\cos t\rangle$ thus
$\mathbf{r}'(t) = \langle 2e^{2t}, \cos t - t\sin t\rangle$. Therefore
$\mathbf{r}'(0) = \langle 2e^0, \cos(0) - (0)\sin(0)\rangle = \langle 2, 1\rangle$. The unit vector is $\dfrac{\langle 2,1\rangle}{|\langle 2,1\rangle|} = \left\langle \dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right\rangle$.

9. C: By the chain rule, if $h(x) = f(g(x))$,
$h'(x) = f'(g(x))g'(x)$.
Thus $h'(-3) = f'(g(-3))g'(-3) = f'(-2)(4) = 44$

10. B $\lim_{\theta\to 0} \dfrac{\sin^2(3\theta)}{\theta^2} = \lim_{\theta\to 0}\left(\dfrac{\sin(3\theta)}{\theta}\right)\left(\dfrac{\sin(3\theta)}{\theta}\right)$
$= 9\lim_{\theta\to 0}\left(\dfrac{\sin(3\theta)}{3\theta}\right)\left(\dfrac{\sin(3\theta)}{3\theta}\right) = 9(1)(1) = 9$

11. (i) $f(x) = \tan(2x^3) + \tan^3(x)$. By the chain rule,
$f'(x) = \sec^2(2x^3)(6x^2) + 3\tan^2 x\sec^2 x$.
(ii) $g(t) = \sqrt{1+\sqrt{t}} = (1+t^{1/2})^{1/2}$. By the chain rule,
$g'(t) = \dfrac{1}{2}(1+t^{1/2})^{-1/2}\dfrac{1}{2}t^{-1/2}$
$= \dfrac{1}{4\sqrt{t}\sqrt{1+\sqrt{t}}}$

12. We are given $\dfrac{dV}{dt} = \dfrac{5}{2}$. We want to find $\dfrac{dh}{dt}$ when $h = 2$. The volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$. To get V in terms of h only, we will use similar triangles. $\dfrac{r}{h} = \dfrac{2}{6}$, thus $r = \dfrac{1}{3}h$. Substitute this in for r:
$V = \dfrac{1}{3}\pi(\dfrac{1}{3}h)^2 h = \dfrac{\pi}{27}h^3$. Differentiate with respect to time: $\dfrac{dV}{dt} = \dfrac{\pi}{9}h^2\dfrac{dh}{dt}$. Now substitute $\dfrac{dV}{dt} = \dfrac{5}{2}$ and $h = 2$ yields $\dfrac{5}{2} = \dfrac{\pi}{9}(2)^2\dfrac{dh}{dt}$, thus $\dfrac{dh}{dt} = \dfrac{45}{8\pi}$ inches per second.

13. First we will differentiate implicitly with respect to x:

$y^2 \sin 2x = 8 - 2y$. Using the product rule and chain rule,

$2y\frac{dy}{dx}\sin 2x + y^2 2\cos 2x = -2\frac{dy}{dx}$. Solve for $\frac{dy}{dx}$.

$\frac{dy}{dx}(2y\sin 2x + 2) = -2y^2\cos 2x$, thus

$\frac{dy}{dx} = \frac{-2y^2\cos 2x}{2y\sin 2x + 2}$. To find the slope of the tangent line, we will substitute the point $\left(\frac{\pi}{4}, 2\right)$ into $\frac{dy}{dx}$.

$m = \frac{-2(2)^2\cos\frac{\pi}{2}}{2(2)\sin(\frac{\pi}{2}) + 2} = 0$. Thus the equation of the line is $y - 2 = 0(x - \pi/4)$, hence $y = 2$.

14. (i) To find the tangent line, we need a point and the slope. To find the point, substitute $t = 1$ into the parametric equations:

$x = t^3 - 6t^2$, $y = t^2 - 6t$, so when $t = 1$, $x = -5$ and $y = -5$. Thus the point is $(-5, -5)$. Now, the slope of the tangent line is

$m = \frac{dy/dt}{dx/dt}$ evaluated at $t = 1$. $\frac{dy}{dx} = \frac{2t - 6}{3t^2 - 12t}$, thus when $t = 1$, $m = \frac{4}{9}$. So the equation of the tangent line is $y + 5 = \frac{4}{9}(x + 5)$.

(ii) The tangent line is vertical where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. This happens when $3t^2 - 12t = 0$, thus $t = 0$ or $t = 4$. Substitute $t = 0$ and $t = 4$ into $x = t^3 - 6t^2$, $y = t^2 - 6t$, we get the points $(0, 0)$ and $(-32, -8)$.

(iii) The tangent line is horizontal where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. This happens when $2t - 6 = 0$, thus $t = 3$. Substitute this into $x = t^3 - 6t^2$, $y = t^2 - 6t$, we get the point $(-27, -9)$.

15. Using differentials, $f(a + dx) \approx f(a) + f'(a)dx$, where $f(x) = \sqrt{x}$, $a = 9$ and $dx = 0.02$. Thus

$f(9.02) \approx f(9) + f'(9)(0.02)$

$= 3 + \frac{1}{6}(0.02) = \frac{901}{300}$

Using Linear Approximation: Find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

$L(x) = f(9) + f'(9)(x - 9)$.

Now, $f(9) = \sqrt{9} = 3$ and $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

Hence $L(x) = 3 + \frac{1}{6}(x - 9)$. Now, to approximate $\sqrt{9.02}$, substitute $x = 9.02$ in $L(x)$ for x:

$L(9.02) = 3 + \frac{1}{6}(9.02 - 9) = 3 + \frac{1}{6}(0.02) = \frac{901}{300}$

16. To find where the tangent line is horizontal, we need to solve $f'(x) = 0$ for $0 \leq x \leq 2\pi$. $f(x) = x + 2\sin x$, thus $f'(x) = 1 + 2\cos x$. Solve $f'(x) = 0$ gives $\cos x = -\frac{1}{2}$. For $0 \leq x \leq 2\pi$, this happens when $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.