Solutions to Exam III Form A

1. **D** By the definition of logarithmic functions, $x + 2 = 2^3 = 8$, so $x = 6$.

2. **E** The domain is all x such that $-1 \le x - 2 \le 1$, which means $1 \le x \le 3$.

3. **C** The derivative is $y' = 1 - \dfrac{1}{1+x^2}$, so the slope is $y'(1) = 1 - \dfrac{1}{1+1^2} = \dfrac{1}{2}$.

4. **C** Use logarithmic differentiation:
   $\ln y = \ln(x^{\tan x}) = \tan x \ \ln x$
   $\dfrac{y'}{y} = \tan x \left(\dfrac{1}{x}\right) + \sec^2 x \ln x$
   $y' = x^{\tan x} \left(\dfrac{\tan x}{x} + \ln x \ \sec^2 x\right)$

5. **E** As $x \to 0^+, \ln x \to -\infty$ and $x^2 \to 0$, so the limit is either ∞ or $-\infty$. Since the numerator is negative (approaching $-\infty$) and the denominator is positive, the limit is $-\infty$. (NOTE that since the limit is in a determinite form, we should NOT use L'Hospital's Rule)

6. **D** The limit is of the indeterminite form $\dfrac{0}{0}$, so we do use L'Hospital's Rule this time:
   $\lim\limits_{x \to 0} \dfrac{\sin x + x \cos x}{-\sin x}$.This is still of the form $\dfrac{0}{0}$, so we can either use L'Hospital's Rule again, or note that the expression simplifies to
   $\lim\limits_{x \to 0} -1 - \dfrac{x}{\sin x}(\cos x) = -1 - (1)(\cos 0) = -1 - 1 = -2$.

7. **C** Assuming exponential growth, the equation is $y = Ce^{kt}$. When $t = 0$, $y = 500$, so $500 = Ce^{k(0)}$, or $C = 500$. When $t = 3$, $y = 8000$, so $8000 = 500e^{3k}$. Solving for k yields $k = \dfrac{1}{3} \ln 16$ (NOTE: you can also solve using $e^k = 16^{1/3}$. Therefore, the expression is $y = 500e^{(1/3 \ln 16)t} = 500e^{\ln(16^{t/3})} = 500(16)^{t/3}$.

8. **B** Find the second derivative and equate to 0:
   $y' = \dfrac{x}{2} + \cos x,\ y'' = \dfrac{1}{2} - \sin x = 0$
   $\sin x = \dfrac{1}{2}$, or $x = \dfrac{\pi}{6}$and $x = \dfrac{5\pi}{6}$(since the interval is $0 \le x \le \pi$). Testing f'' at a value in each of the subintervals yields $f'' > 0$ when $0 \le x < \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6} < x \le \pi$and $f'' < 0$ when $\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$. Therefore, there are points of inflection when $x = \dfrac{\pi}{6}$ and $x = \dfrac{5\pi}{6}$.

9. **E** Let $y = \sin^{-1} t$. Then $\sin y = t = \dfrac{t}{1}$, which can be illustrated using the right triangle below (since $t > 0$). Therefore, $\tan y = \dfrac{t}{\sqrt{1-t^2}}$.

10. **D** Since f only has one critical number at $x = 1$, $f'(1) = 0$ (since f is differentiable everywhere) and $f'(x) \neq 0$ for all other values of x. To find the critical numbers of h, set $h'(x) = 2xf'(x^2) = 0$. This is true when $x = 0$ and when $x^2 = 1$, or $x = \pm 1$.

11. **D** Dividing by x^2 yields $g'(x) = 2 + \dfrac{3}{x}$, or $g(x) = 2x + 3\ln x + C$. Since $g(e) = 1$, $1 = 2e + 3\ln e + C$, or $C = -2 - 2e$. Therefore, $g(x) = 2x + 3\ln x - (2 + 2e)$.

12. .

(a) The domain of H is all x such that $\sin^{-1} x > 0$ (meaning $x > 0$) and $-1 \leq x \leq 1$. Both are true when $0 < x \leq 1$.

(b) Using the Chain Rule, $H'(x) = \dfrac{1}{\sin^{-1} x} \cdot \dfrac{1}{\sqrt{1-x^2}}$.

(c) The domain of the derivative must be a subset of $(0,1]$ (the domain of the original function). The only value in this interval where $f'(x)$ is not defined is $x = 1$. Therefore, the domain of the derivative is $0 < x < 1$.

13. .

(a) Using the product rule, $f'(x) = e^{-2x^2} + xe^{-2x^2}(-4x) = e^{-2x^2}(1-4x^2)$. Differentiate this using the product rule again yields $f''(x) = e^{-2x^2}(-4x)(1-4x^2) + e^{-2x^2}(-8x) = -4xe^{-2x^2}(1-4x^2+2) = -4xe^{-2x^2}(3-4x^2)$

(b) The critical numbers occur when $f' = 0$. Since $e^{-2x^2} \neq 0$ for all x, the solution is $1-4x^2 = 0$, or $x = \pm\dfrac{1}{2}$.

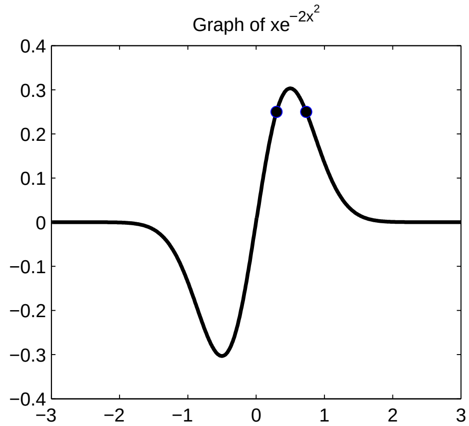
(c) The critical numbers divide the x-axis into three subintervals. Test a value from each subinterval by substituting into f'. $f'(x) < 0$ when $x < -\dfrac{1}{2}$ and when $x > \dfrac{1}{2}$, and $f'(x) > 0$ when $-\dfrac{1}{2} < x < \dfrac{1}{2}$. Therefore, f is increasing on $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and f is decreasing on $\left(-\infty, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$.

(d) From part (c), the local maximum is $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}e^{-1/2}$.

(e) $f\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}e^{-1/2}$. Since $f(x) \to 0$ as $x \to \pm\infty$, the absolute maximum is $\dfrac{1}{2}e^{-1/2}$ and the absolute minimum is $f\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}e^{-1/2}$.

(f) $f''(x) = 0$ when $x = 0$, $x = \pm\dfrac{\sqrt{3}}{2}$. Test a value from each of the four subintervals by substituting into f''. $f''(x) < 0$ when $x < -\dfrac{\sqrt{3}}{2}$ and when $0 < x < \dfrac{\sqrt{3}}{2}$. $f''(x) > 0$ when $-\dfrac{\sqrt{3}}{2} < x < 0$ and when $x > \dfrac{\sqrt{3}}{2}$. Therefore, f is concave up on $\left(-\dfrac{\sqrt{3}}{2}, 0\right) \cup \left(\dfrac{\sqrt{3}}{2}, \infty\right)$.

(g) The absolute maximum of f is $\dfrac{1}{2}e^{-1/2} = \dfrac{1}{2\sqrt{e}}$, which is greater than $\dfrac{1}{4}$. Therefore, from the graph of f shown below (determined by information in the previous parts), there are two values of x where $f(x) = \dfrac{1}{4}$.



14. Let $y = \lim_{x\to\infty} \left(\frac{x-3}{x+4}\right)^x$. Then $\ln y = \lim_{x\to\infty} x \ln\left(\frac{x-3}{x+4}\right) = \lim_{x\to\infty} \frac{\ln(\frac{x-3}{x+4})}{\frac{1}{x}}$. Applying L'Hospital's Rule yields $\lim_{x\to\infty} \frac{(\frac{x+4}{x-3}) \cdot \frac{(x+4)(1)-(x-3)(1)}{(x+4)^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x+4}{x-3} \cdot \frac{7}{(x+4)^2} \cdot (-x^2) = (1)(-7) = -7$. Since $\ln y \to -7$, $y \to e^{-7}$.

15. Our goal is to maximize $S = 2\pi r^2 + 2\pi r h$ with the condition that $V = \pi r^2 h = 22$. Then $h = \frac{22}{\pi r^2}$. Substitution into S yields $S = 2\pi r^2 + 2\pi r\left(\frac{22}{\pi r^2}\right) = 2\pi r^2 + \frac{44}{r}$. Differentiate to find the critical values: $S' = 4\pi r - \frac{44}{r^2} = 0$, $4\pi r^3 - 44 = 0$, $r = \sqrt[3]{\frac{11}{\pi}}$. Show this critical value is a minimum by showing S is decreasing, then increasing, or showing that $S'' = 4\pi + \frac{88}{r^3} > 0$. The dimensions of the can are $r = \sqrt[3]{\frac{11}{\pi}}$ inches and $h = \frac{22}{\pi(\frac{11}{\pi})^{2/3}} = 2\sqrt[3]{\frac{11}{\pi}}$ inches.