Solutions to Exam III Form B

1. **A** By the definition of logarithmic functions, $x - 2 = 2^3 = 8$, so $x = 10$.

2. **B** The domain is all x such that $-1 \leq x + 2 \leq 1$, which means $-3 \leq x \leq -1$.

3. **D** The derivative is $y' = \dfrac{1}{1+x^2} - 1$, so the slope is $y'(1) = \dfrac{1}{1+1^2} - 1 = -\dfrac{1}{2}$.

4. **B** Use logarithmic differentiation:
$\ln y = \ln(x^{\sec x}) = \sec x \ln x$
$\dfrac{y'}{y} = \sec x \left(\dfrac{1}{x}\right) + \sec x \tan x \ln x$
$y' = x^{\sec x}\left(\dfrac{\sec x}{x} + \ln x \sec x \tan x\right)$

5. **B** As $x \to 0^+, \ln x \to -\infty$ and $x^2 \to 0$, so the limit is either ∞ or $-\infty$. Since the numerator is negative (approaching $-\infty$) and the denominator is positive, the limit is $-\infty$. (NOTE that since the limit is in a determinite form, we should NOT use L'Hospital's Rule)

6. **C** The limit is of the indeterminite form $\dfrac{0}{0}$, so we do use L'Hospital's Rule this time:
$\lim\limits_{x\to 0} \dfrac{\sin x + x\cos x}{\sin x}$.This is still of the form $\dfrac{0}{0}$, so we can either use L'Hospital's Rule again, or note that the expression simplifies to
$\lim\limits_{x\to 0} 1 + \dfrac{x}{\sin x}(\cos x) = 1 + (1)(\cos 0) = 1 + 1 = 2$.

7. **D** Assuming exponential growth, the equation is $y = Ce^{kt}$. When $t = 0$, $y = 500$, so $500 = Ce^{k(0)}$, or $C = 500$. When $t = 3$, $y = 9000$, so $9000 = 500e^{3k}$. Solving for k yields $k = \dfrac{1}{3}\ln 18$ (NOTE: you can also solve using $e^k = 18^{1/3}$. Therefore, the expression is $y = 500e^{(1/3 \ln 18)t} = 500e^{\ln(18^{t/3})} = 500(18)^{t/3}$.

8. **C** Find the second derivative and equate to 0:
$y' = \dfrac{\sqrt{3}\,x}{2} + \cos x,\ y'' = \dfrac{\sqrt{3}}{2} - \sin x = 0$
$\sin x = \dfrac{\sqrt{3}}{2}$, or $x = \dfrac{\pi}{3}$and $x = \dfrac{2\pi}{3}$(since the interval is $0 \leq x \leq \pi$). Testing f'' at a value in each of the subintervals yields $f'' > 0$ when $0 \leq x < \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3} < x \leq \pi$and $f'' < 0$ when $\dfrac{\pi}{3} < x < \dfrac{2\pi}{3}$. Therefore, there are points of inflection when $x = \dfrac{\pi}{3}$ and $x = \dfrac{2\pi}{3}$.

9. **B** Let $y = \cos^{-1} t$. Then $\cos y = t = \dfrac{t}{1}$, which can be illustrated using the right triangle below (since $t > 0$). Therefore, $\tan y = \dfrac{\sqrt{1-t^2}}{t}$.

10. **A** Since f only has one critical number at $x = 1$, $f'(1) = 0$ (since f is differentiable everywhere) and $f'(x) \neq 0$ for all other values of x. To find the critical numbers of h, set $h'(x) = 2xf'(x^2) = 0$. This is true when $x = 0$ and when $x^2 = 1$, or $x = \pm 1$.

11. **B** Dividing by x^2 yields $g'(x) = 3 + \dfrac{2}{x}$, or $g(x) = 3x + 2\ln x + C$. Since $g(e) = 1$, $1 = 3e + 2\ln e + C$, or $C = -1 - 3e$. Therefore, $g(x) = 2x + 3\ln x - (1 + 3e)$.

12. .

(a) The domain of H is all x such that $\sin^{-1} x > 0$ (meaning $x > 0$) and $-1 \le x \le 1$. Both are true when $0 < x \le 1$.

(b) Using the Chain Rule, $H'(x) = \dfrac{1}{\sin^{-1} x} \cdot \dfrac{1}{\sqrt{1-x^2}}$.

(c) The domain of the derivative must be a subset of $(0, 1]$ (the domain of the original function). The only value in this interval where $f'(x)$ is not defined is $x = 1$. Therefore, the domain of the derivative is $0 < x < 1$.

13. .

(a) Using the product rule, $f'(x) = e^{-2x^2} + xe^{-2x^2}(-4x) = e^{-2x^2}(1 - 4x^2)$. Differentiate this using the product rule again yields $f''(x) = e^{-2x^2}(-4x)(1 - 4x^2) + e^{-2x^2}(-8x) = -4xe^{-2x^2}(1 - 4x^2 + 2) = -4xe^{-2x^2}(3 - 4x^2)$

(b) The critical numbers occur when $f' = 0$. Since $e^{-2x^2} \ne 0$ for all x, the solution is $1-4x^2 = 0$, or $x = \pm\dfrac{1}{2}$.

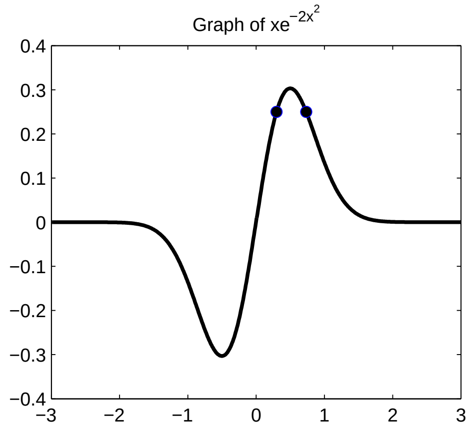
(c) The critical numbers divide the x-axis into three subintervals. Test a value from each subinterval by substituting into f'. $f'(x) < 0$ when $x < -\dfrac{1}{2}$ and when $x > \dfrac{1}{2}$, and $f'(x) > 0$ when $-\dfrac{1}{2} < x < \dfrac{1}{2}$. Therefore, f is increasing on $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and f is decreasing on $\left(-\infty, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$.

(d) From part (c), the local minimum is $f\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}e^{-1/2}$.

(e) $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}e^{-1/2}$. Since $f(x) \to 0$ as $x \to \pm\infty$, the absolute maximum is $\dfrac{1}{2}e^{-1/2}$ and the absolute minimum is $f\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}e^{-1/2}$.

(f) $f''(x) = 0$ when $x = 0$, $x = \pm\dfrac{\sqrt{3}}{2}$. Test a value from each of the four subintervals by substituting into f''. $f''(x) < 0$ when $x < -\dfrac{\sqrt{3}}{2}$ and when $0 < x < \dfrac{\sqrt{3}}{2}$. $f''(x) > 0$ when $-\dfrac{\sqrt{3}}{2} < x < 0$ and when $x > \dfrac{\sqrt{3}}{2}$. Therefore, f is concave down on $\left(-\infty, -\dfrac{\sqrt{3}}{2}\right) \cup \left(0, \dfrac{\sqrt{3}}{2}\right)$.

(g) The absolute maximum of f is $\dfrac{1}{2}e^{-1/2} = \dfrac{1}{2\sqrt{e}}$, which is greater than $\dfrac{1}{4}$. Therefore, from the graph of f shown below (determined by information in the previous parts), there are two values of x where $f(x) = \dfrac{1}{4}$.



14. Let $y = \lim_{x\to\infty} \left(\frac{x-3}{x+5}\right)^x$. Then $\ln y = \lim_{x\to\infty} x \ln\left(\frac{x-3}{x+5}\right) = \lim_{x\to\infty} \frac{\ln(\frac{x-3}{x+5})}{\frac{1}{x}}$. Applying L'Hospital's Rule yields $\lim_{x\to\infty} \frac{(\frac{x+5}{x-3}) \cdot \frac{(x+5)(1)-(x-3)(1)}{(x+5)^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x+5}{x-3} \cdot \frac{8}{(x+5)^2} \cdot (-x^2) = (1)(-8) = -8$. Since $\ln y \to -8$, $y \to e^{-8}$.

15. Our goal is to maximize $S = 2\pi r^2 + 2\pi rh$ with the condition that $V = \pi r^2 h = 24$. Then $h = \frac{24}{\pi r^2}$. Substitution into S yields $S = 2\pi r^2 + 2\pi r\left(\frac{24}{\pi r^2}\right) = 2\pi r^2 + \frac{48}{r}$. Differentiate to find the critical values: $S' = 4\pi r - \frac{48}{r^2} = 0$, $4\pi r^3 - 48 = 0$, $r = ^3\sqrt{\frac{12}{\pi}}$. Show this critical value is a minimum by showing S is decreasing, then increasing, or showing that $S'' = 4\pi + \frac{96}{r^3} > 0$. The dimensions of the can are $r = ^3\sqrt{\frac{12}{\pi}}$inches and $h = \frac{24}{\pi(\frac{12}{\pi})^{2/3}} = 2\,^3\sqrt{\frac{12}{\pi}}$inches.