Math 151 Fall 2010 Exam I Solutions-Form A

1. C: Find the vector $\vec{a}$ that has magnitude $|\vec{a}| = 12$ and makes an angle of $300°$ with the positive x-axis. Since $300° = 360° - 60°$, our reference angle is $60°$ in quadrant IV. Thus

   $\vec{a} = \langle 12\cos 60°, -12\sin 60°\rangle = \left\langle 6, -6\sqrt{3}\right\rangle$.

2. E: $\lim\limits_{t\to\infty} \dfrac{6t^2+3t}{(3-t)(2t+4)} = \lim\limits_{t\to\infty} \dfrac{6t^2+3t}{-2t^2+2t+12}$

   $= \lim\limits_{t\to\infty} \dfrac{6+3/t}{-2+2/t+12/t^2} = -3$

3. A: Find all vertical asymptotes for $f(x) = \dfrac{x^2-1}{x^3+x^2}$. First by simplifying $f(x)$, we find that

   $f(x) = \dfrac{(x+1)(x-1)}{x^2(x+1)} = \dfrac{x-1}{x^2}$, provided $x \neq -1$ and $x \neq 0$. Thus the only vertical asymptote occurs at $x = 0$.

4. B: A force $\vec{F} = 2\vec{i} + 6\vec{j}$ moves an object from the point $P(1,3)$ to the point $Q(3,7)$. How much work is done if the force is measured in pounds and the distance is measured in feet? Here, $\vec{F} = \langle 2,6\rangle$ and $\vec{D} = \overrightarrow{PQ} = \langle 2,4\rangle$. Hence the work done is

   $W = \vec{F}\cdot\vec{D} = \langle 2,6\rangle \cdot \langle 2,4\rangle = 28$ foot pounds.

5. E: Find the equation of the tangent line to the graph of $f(x) = \dfrac{x}{1+2x}$ at $x = 1$. The slope of the tangent line at $x = 1$ is $m = f'(1)$. Using the quotient rule, we find that $f'(x) = \dfrac{1}{(1+2x)^2}$. Thus $m = \dfrac{1}{9}$. The line is tangent to the curve at the point $(1, \dfrac{1}{3})$. Thus the equation of the tangent line is $y - \dfrac{1}{3} = \dfrac{1}{9}(x-1)$.

6. A: The parametric equations $x = 3 + \sin t$, $y = -2 + \cos t$ describe a circle since $x - 3 = \sin t$ and $y + 2 = \cos t$ and therefore $(x-3)^2 + (y+2)^2 = 1$.

7. E: Find the value of a and b that makes

   $f(x) = \begin{cases} -x + a & \text{if } x \leq 1 \\ bx^2 + 3 & \text{if } x > 1 \end{cases}$ continuous and differentiable at $x = 1$. For f to be continuous at $x = 1$, we need for $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x)$. Thus $b+3 = -1+a$. For f to be differentiable at $x = 1$, we need for $\lim\limits_{x\to 1^+} f'(x) = \lim\limits_{x\to 1^-} f'(x)$. Thus $2b = -1$. This yields $b = -\dfrac{1}{2}$ and therefore $a = \dfrac{7}{2}$.

8. B: What is the slope of the line that passes through the point $(4,7)$ and is perpendicular to the vector $\langle -5,6\rangle$? The direction vector is $\langle -5,6\rangle^{\perp} = \langle -6,-5\rangle$. Thus the slope of the line with direction vector $\langle -6,-5\rangle$ is $\dfrac{5}{6}$.

9. A: Which of the following intervals contains a solution to the equation $x^3 + 2x + 2 = 7$? First, note $f(x) = x^3+2x+2$ is continuous, so we will apply the Intermediate Value Theorem. Since $f(1) = 5 < 7$ and $f(2) = 14 > 7$, there is a solution to $f(c) = 7$ on the interval $[1,2]$.

10. B: If ΔABC is an equilateral triangle with sides of length 2, compute $\overrightarrow{BA}\cdot\overrightarrow{BC}$.

    $\overrightarrow{BA}\cdot\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\theta$. We are given $|\overrightarrow{BA}| = 2$, $|\overrightarrow{BC}| = 2$, and since the triangle is equilateral, $\theta = 60°$. Hence $\overrightarrow{BA}\cdot\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\theta = (2)(2)\cos 60° = 2$.

11. Find the distance from the point $P(1,7)$ to the line $y = 2x + 3$. First we will choose any two points on the line.

    Let $A(0,3)$ and $B(1,5)$. Define $\vec{a} = \overrightarrow{AB} = \langle 1,2\rangle$ and $\vec{b} = \overrightarrow{AP} = \langle 1,4\rangle$. Then the distance from the point $P(1,7)$ to the line $y = 2x+3$ is

    $d = comp_{a^{\perp}}\vec{b} = \dfrac{\vec{a}^{\perp}\cdot\vec{b}}{|\vec{a}^{\perp}|} = \dfrac{\langle -2,1\rangle\cdot\langle 1,4\rangle}{|\langle -2,1\rangle|} = \dfrac{2}{\sqrt{5}}$

12. Find the coordinates of the intersection point of the lines L1 and L2 given below.

    L1: $x = -1 - 2t$, $y = 2 + t$

    L2: $x = 15 - 3w$, $y = 3 + 6w$

    We find the intersection of the lines by solving the equations

    (i) $-1 - 2t = 15 - 3w$ and (ii) $2 + t = 3 + 6w$.

    Solving (ii) for t yields $t = 1+6w$. Substituting this into (i) for t yields $-1-2(1+6w) = 15-3w$. Solving this for w gives $w = -2$, and therefore $t = -11$. Now to find the intersection point, we can either substitute $t = -11$ into L1 or substitute $w = -2$ into L2. Both yield the point $(21,-9)$.

13. For each of the following limits, either calculate the limit, if it exists, or else explain why the limit does not exist.

(i) $\lim_{x\to 0^+} \frac{x^2+2x}{|x|} = \lim_{x\to 0^+} \frac{x(x+2)}{x} = 2$

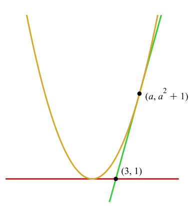
(ii) $\lim_{x\to 0^-} \frac{x^2+2x}{|x|} = \lim_{x\to 0^-} \frac{x(x+2)}{-x} = -2$

(iii) $\lim_{x\to 0} \frac{x^2+2x}{|x|}$ does not exist since the left and right hand limits at 0 are not equal.

14. Using the *definition of the derivative*, find $f'(x)$ for $f(x) = \frac{4}{x+1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4}{x+h+1} - \frac{4}{x+1}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4}{x+h+1} - \frac{4}{x+1}}{h} \frac{(x+h+1)(x+1)}{(x+h+1)(x+1)}$$
$$= \lim_{h\to 0} \frac{4(x+1)-4(x+h+1)}{h(x+h+1)(x+1)}$$
$$= \lim_{h\to 0} \frac{-4h}{h(x+h+1)(x+1)}$$
$$= \frac{-4}{(x+1)^2}$$

15. The horizontal line through the point $(3,1)$ is tangent to the parabola $y = x^2+1$. We see from the figure below that there is a second line tangent to the parabola at (a, a^2+1) that also passes through the point $(3,1)$. Find the value of a.



The slope of the tangent line to the curve $y = x^2+1$ at $x = a$ is $m_1 = 2a$. This tangent line can also be viewed as a line passing through the points $(3,1)$ and (a, a^2+1), and therefore the slope is also

$m_2 = \frac{a^2+1-1}{a-3}$. Thus to find a we will solve $m_1 = m_2$:

$2a = \frac{a^2}{a-3} \Rightarrow 2a(a-3) = a^2 \Rightarrow 2a^2 - 6a = a^2$, thus $a = 0$ or $a = 6$. Clearly from the picture provided, $a > 0$, hence $a = 6$.

16. (a) Find $\lim_{x\to 2} \frac{x-\sqrt{3x-2}}{x^2-4}$

$$= \lim_{x\to 2} \frac{(x-\sqrt{3x-2})(x+\sqrt{3x-2})}{(x^2-4)(x+\sqrt{3x-2})}$$
$$= \lim_{x\to 2} \frac{x^2-(3x-2)}{(x+2)(x-2)(x+\sqrt{3x-2})}$$
$$= \lim_{x\to 2} \frac{(x-2)(x-1)}{(x+2)(x-2)(x+\sqrt{3x-2})}$$
$$= \lim_{x\to 2} \frac{x-1}{(x+2)(x+\sqrt{3x-2})} = \frac{1}{16}$$

(b) Is there a value of a for which $\lim_{x\to -3} \frac{x^2+ax+a+5}{x^2-2x-15}$ exists? If so, find the value of a. If not, explain why.

Note that $\lim_{x\to -3} \frac{x^2+ax+a+5}{x^2-2x-15} = \lim_{x\to -3} \frac{x^2+ax+a+5}{(x-5)(x+3)}$. Hence in order for this limit to exist, the numerator must have a factor of $(x+3)$ (thereby eliminating the division by zero). Thus the numerator evaluated at $x = -3$ must be zero:

$(-3)^2 + a(-3) + a + 5 = 0 \Rightarrow 14 - 2a = 0$, thus $a = 7$.