Math 151 Fall 2010 Exam II Solutions-Form A

1. B. $\lim_{t\to0}\dfrac{t^3}{\sin^3(2t)} = \lim_{t\to0}\left(\dfrac{t}{\sin(2t)}\right)^3$
$= \dfrac{1}{8}\lim_{t\to0}\left(\dfrac{2t}{\sin(2t)}\right)^3.$
Now, since $\lim_{\theta\to0}\dfrac{\sin\theta}{\theta} = 1$, it follows that $\lim_{\theta\to0}\dfrac{\theta}{\sin\theta} = 1$.
Thus $\dfrac{1}{8}\lim_{t\to0}\left(\dfrac{2t}{\sin(2t)}\right)^3 = \dfrac{1}{8}(1)^3 = \dfrac{1}{8}$

2. C. Find $f'\left(\dfrac{\pi}{4}\right)$ for $f(x) = x\sin x\cos x$. By the product rule, $f'(x) = \sin x\cos x + x(\cos^2 x - \sin^2 x)$. Thus
$f'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\dfrac{\sqrt{2}}{2} + \dfrac{\pi}{4}(\dfrac{1}{2} - \dfrac{1}{2}) = \dfrac{1}{2}$

3. A. If $g(x) = f(x^3) + (f(x))^3$, $f(1) = 2$, $f'(1) = -3$, find $g'(1)$. By the chain rule,
$g'(x) = f'(x^3)3x^2 + 3(f(x))^2 f'(x)$.
Thus $g'(1) = f'(1)3(1)^2 + 3(f(1))^2 f'(1)$
$= (-3)(3) + 3(4)(-3) = -45$

4. E. Find the slope of the parametric curve $x = e^{-5t}$, $y = t\cos t$ at the point $(1, 0)$. Note first that $t = 0$ yields the point $(1, 0)$. Thus the slope of the tangent line is
$m = \dfrac{dy/dt}{dx/dt}$ evaluated at $t = 0$.
$m = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t - t\sin t}{-5e^{-5t}}$. Substitute $t = 0$ yields $m = -\dfrac{1}{5}$

5. E. If $g(x)$ is the inverse of $f(x) = \sqrt{x^3 + x + 6}$, find $g'(4)$. We know that $g'(4) = \dfrac{1}{f'(g(4))}$. Now, since $f(2) = 4$, it follows that $g(4) = 2$. Hence,
$g'(4) = \dfrac{1}{f'(g(4))} = \dfrac{1}{f'(2)}$.
Now, $f'(x) = \dfrac{1}{2}(x^3 + x + 6)^{-1/2}(3x^2 + 1)$.
Thus, $f'(2) = \dfrac{1}{2}(16)^{-1/2}(13) = \dfrac{13}{8}$. Hence,
$g'(4) = \dfrac{1}{f'(g(4))} = \dfrac{1}{f'(2)} = \dfrac{8}{13}$

6. B. The graph of the curve $y = x + \dfrac{1}{3}\cos 3x$ has a horizontal tangent at $x =?$. $f(x)$ has a horizontal tangent where $f'(x) = 0$. Thus $1 - \sin(3x) = 0$, hence $\sin(3x) = 1$ yielding $x = \dfrac{\pi}{6}$.

7. D. An object is moving according to the equation $s(t) = t^3 - 9t^2 + 15t + 8$. When is the object moving in the negative direction? The object is moving in the negative direction when $v(t) < 0$. Now,
$v(t) = s'(t) = 3t^2 - 18t + 15 = 3(t - 5)(t - 1)$.
Solve $v(t) < 0$ yields $1 < t < 5$.

8. D. $\lim_{x\to4^-}\left(\dfrac{1}{3}\right)^{\frac{x}{x-4}} = \left(\dfrac{1}{3}\right)^{\lim_{x\to4^-}\frac{x}{x-4}} = \left(\dfrac{1}{3}\right)^{-\infty} = \infty.$

9. A. Find the slope of the tangent line to the curve $y^3 - xy = 2x + 4$ at the point $(1, 2)$. Differentiating implicitly, $3y^2\dfrac{dy}{dx} - y - x\dfrac{dy}{dx} = 2$. Solve for $\dfrac{dy}{dx}$ yields $\dfrac{dy}{dx} = \dfrac{2 + y}{3y^2 - x}$. Substitute $x = 1$ and $y = 2$ gives us a slope of $m = \dfrac{4}{11}$.

10. A. Find the linear approximation, $L(x)$, of $f(x) = \sqrt{x^2 + 9}$ at $x = -4$. The linear approximation for $f(x)$ at $x = -4$ is
$L(x) = f(-4) + f'(-4)(x - (-4))$.
$f(-4) = \sqrt{25} = 5$. Since $f'(x) = \dfrac{x}{\sqrt{x^2 + 9}}$, it follows that $f'(-4) = \dfrac{-4}{5}$. Thus
$L(x) = 5 - \dfrac{4}{5}(x + 4)$ or $L(x) = \dfrac{9}{5} - \dfrac{4}{5}x$.

11. D. Find a unit tangent vector to the curve $\mathbf{r}(t) = \left\langle \dfrac{2}{t} + 1, t\sqrt{t} + \dfrac{t}{2} \right\rangle$ at $t = 1$.
$\mathbf{r}'(t) = \left\langle -\dfrac{2}{t^2}, \dfrac{3}{2}t^{1/2} + \dfrac{1}{2} \right\rangle$. Thus the tangent vector at $t = 1$ is $r'(1) = \langle -2, 2\rangle$. Divide by the magnitude to make the tangent vector a unit vector yields
$\dfrac{\langle -2, 2\rangle}{\sqrt{8}} = \left\langle \dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$

12. Given that $s(t) = e^{-3t}\cos(2t)$ is the equation of motion of an object, find
(i) The velocity at time t. By the product and chain rule,
$v(t) = s'(t) = -3e^{-3t}\cos(2t) + e^{-3t}(-2\sin(2t))$
$= -e^{-3t}(3\cos(2t) + 2\sin(2t))$.
(ii) The acceleration at time t. $a(t) = v'(t) = s''(t)$
$= 3e^{-3t}(3\cos(2t) + 2\sin(2t)) - e^{-3t}(-6\sin(2t) + 4\cos(2t))$

13. Find the equation of the tangent line to the curve

$y = \sqrt{x + \sqrt{8+x}}$ at $x = 1$. The point of tangency is $(1, 2)$. The slope of the tangent line is found by taking the derivative of y and then substituting $x = 1$.

$y = (x + (8 + x)^{1/2})^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}\left(x + (8+x)^{1/2}\right)^{-1/2}\left(1 + \frac{1}{2}(8+x)^{-1/2}\right).$$

When $x = 1$, $\frac{dy}{dx} = \frac{7}{24}$. Thus the equation of the tangent line is

$y - 2 = \frac{7}{24}(x - 1)$ or $y = \frac{7}{24}x + \frac{41}{24}$.

14. Use differentials (or a linear approximation) to estimate $(1.02)^{10}$. Let $f(x) = x^{10}$, $a = 1$ and therefore $dx = 0.02$. Using differentials,

$f(1.02) = (1.02)^{10} \approx f(1) + 10(0.02)$

$= 1 + 10(0.02) = 1.2$

15. The volume of a cube is increasing at a rate of 10 cubic centimeters per minute. How fast is the surface area increasing when the length of the edge is 30 cm?

The volume of a cube is $V = x^3$. The surface area is $A = 6x^2$. We want to express the surface area in terms of the volume. Solve $V = x^3$ for x yields $x = \sqrt[3]{V}$.

Thus, $A = 6V^{2/3}$. $\frac{dA}{dt} = 4V^{-1/3}\frac{dV}{dt}$. Now, when $x = 30$, $V = 30^3$. Thus $\frac{dA}{dt} = 4(30^3)^{-1/3}(10) = \frac{4}{3}$ square cm per minute.

16. Find $f'(t)$ for the following functions:

(i) $f(t) = \tan^3(t^2 + a^2)$

$f'(t) = 3\tan^2(t^2 + a^2)\sec^2(t^2 + a^2)(2t)$

(ii) $f(t) = e^{t+\sqrt{t}}$

$f'(t) = (1 + \frac{1}{2\sqrt{t}})e^{t+\sqrt{t}}$