Math 151 Fall 2010 Exam 3 Solutions-Form A

1. B. $\lim_{t\to\frac{\pi}{2}^-} \ln(\cos t) = \ln\left(\lim_{t\to\frac{\pi}{2}^-} \cos t\right) = -\infty$,

   since $\cos\left(\frac{\pi}{2}\right) = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$.

2. D. If it is given that $f'(x) = 4 - \frac{5}{x^2+1}$ and $f(1) = 0$, find $f(x)$. Take an antiderivative of $f'(x)$ to obtain $f(x)$: $f(x) = 4x - 5\arctan x + C$.

   Since $f(1) = 0$, $4(1) - 5\arctan(1) + C = 0$. Hence

   $4 - 5\left(\frac{\pi}{4}\right) + C = 0$, yielding $C = \frac{5\pi}{4} - 4$. Thus

   $f(x) = 4x - 5\arctan x + \frac{5\pi}{4} - 4$

3. D. Find $\lim_{x\to 0}(1-4x)^{1/x}$. This is an indeterminate form of the type 1^∞. Let $y = (1-4x)^{1/x}$. Then

   $\ln y = \ln(1-4x)^{1/x}$

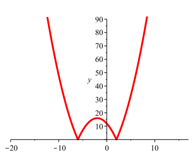
   $= \frac{1}{x}\ln(1-4x) = \frac{\ln(1-4x)}{x}$. Now,

   $\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1-4x)}{x}$ = is of the form $\frac{0}{0}$. Since this is indeterminate form,

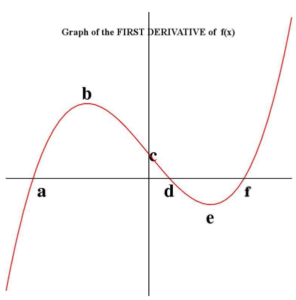
   $\lim_{x\to 0}\frac{\ln(1-4x)}{x} = \lim_{x\to 0}\frac{-4/(1-4x)}{1} = -4$. Hence $\lim_{x\to 0}\ln y = -4$, thus

   $\lim_{x\to 0} y = e^{-4}$.

4. E. The critical numbers are those numbers c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist. By viewing the graph below, $f'(-2) = 0$. Furthermore, $f'(-6)$ and $f'(2)$ do not exist. Thus the critical numbers are $x = -2$, $x = -6$ and $x = 2$.



5. B. The graph of the *DERIVATIVE* of $f(x)$ is shown below. Where does $f(x)$ change concavity?



   $f(x)$ changes concavity when $f'(x)$ goes from increasing to decreasing, that is $x = b$ and $x = e$.

6. D. What is the domain of $f(t) = \arccos(3+t)$? Recall that the domain of $\arccos x$ is the range of $\cos x = [-1, 1]$. Thus the domain of $\arccos(3+t)$ is all t such that

   $-1 \le 3 + t \le 1$, hence $-4 \le t \le -2$.

7. C. Where is $f(x) = e^{x^3-3x}$ increasing? $f(x)$ is increasing where $f'(x) > 0$.

   $f'(x) = (3x^2-3)e^{x^3-3x} = 3(x+1)(x-1)e^{x^3-3x}$. This yields critical numbers of $x = \pm 1$. Now $f'(x) < 0$ on the interval $(-1, 1)$ and $f'(x) > 0$ on the intervals

   $(-\infty, -1) \cup (1, \infty)$.

8. D. Find the derivative of $f(x) = \log_{10}(x^2 + 2^x)$. Recall $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$ and $\frac{d}{dx}a^x = a^x \ln a$. Thus by the chain rule, $\frac{d}{dx}\log_{10}(x^2+2^x) = \frac{2x + 2^x\ln 2}{(x^2+2^x)\ln 10}$

9. B. Where is $f(x) = \ln(x^2+1)$ concave up? $f(x)$ is concave up where $f''(x) > 0$. Now, $f'(x) = \frac{2x}{x^2+1}$ and $f''(x) = \frac{-2x^2+2}{(x^2+1)^2} = \frac{-2(x-1)(x+1)}{(x^2+1)^2}$. Now $f''(x) > 0$ on the interval $(-1, 1)$ and $f''(x) < 0$ on the intervals $(-\infty, -1) \cup (1, \infty)$. Thus $f(x)$ is concave up on the interval $(-1, 1)$.

10. A. Find the absolute extrema for $f(x) = 12 - x - \frac{9}{x}$ on the interval $[1, 4]$. $f'(x) = -1 + \frac{9}{x^2}$. This yields critical numbers of $x = \pm 3$. Since $f(x)$ is only defined on $[1, 4]$, we will only use 3 as a critical number. Now, $f(1) = 2$, $f(4) = \frac{23}{4}$ and $f(3) = 6$. Thus the absolute maximum value is 6 and the absolute minimum value is 2.

11. (i) Find the inverse of the function $f(x) = \frac{3e^{5x-1}}{1+e^{5x-1}}$

    $y = \frac{3e^{5x-1}}{1+e^{5x-1}}$. Interchange x and y: $x = \frac{3e^{5y-1}}{1+e^{5y-1}}$.

    $x(1+e^{5y-1}) = 3e^{5y-1}$

    $x + xe^{5y-1} = 3e^{5y-1}$

    $x = -xe^{5y-1} + 3e^{5y-1}$

    $x = e^{5y-1}(3-x)$.

    $\frac{x}{3-x} = e^{5y-1}$.

    $\ln\left(\frac{x}{3-x}\right) = 5y - 1$, hence

    $y = \frac{1}{5}\left(\ln\frac{x}{3-x} + 1\right)$.

    (ii) Find the domain of the inverse. The domain of $f^{-1}(x) = \frac{1}{5}\left(\ln\frac{x}{3-x} + 1\right)$ is such that $\frac{x}{3-x} > 0$, hence $0 < x < 3$.

12. An object has an acceleration of $\mathbf{a}(t) = t^3\mathbf{i} + 2\sin(t)\mathbf{j}$, an initial velocity of $\mathbf{v}(0) = \mathbf{j}$, and an initial position of $\mathbf{s}(0) = 3\mathbf{i}$. Find a vector function that describes the position of the object at time t.

$\mathbf{a}(t) = t^3\mathbf{i} + 2\sin(t)\mathbf{j}$ thus

$\mathbf{v}(t) = \left(\frac{t^4}{4} + C_1\right)\mathbf{i} + (-2\cos(t) + C_2)\mathbf{j}$. Now since

$\mathbf{v}(0) = \mathbf{j}$, $C_1 = 0$ and $C_2 = 3$. Thus

$\mathbf{v}(t) = \left(\frac{t^4}{4}\right)\mathbf{i} + (-2\cos(t) + 3)\mathbf{j}$.

$\mathbf{s}(t) = \left(\frac{t^5}{20} + C_3\right)\mathbf{i} + (-2\sin(t) + 3t + C_4)\mathbf{j}$. Since

$\mathbf{s}(0) = 3\mathbf{i}$, $C_3 = 3$ and $C_4 = 0$.

$\mathbf{s}(t) = \left(\frac{t^5}{20} + 3\right)\mathbf{i} + (-2\sin t + 3t)\mathbf{j}$

13. If y is a differentiable function of x on the interval $(0, \infty)$, find a formula for $\frac{dy}{dx}$ if $x^y = y^x$.

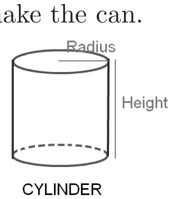
$\ln x^y = \ln y^x$. Thus $y \ln x = x \ln y$. Differentiate implicitly with respect to x:.

$\left(\frac{dy}{dx}\right)\ln x + y\left(\frac{1}{x}\right) = \ln y + x\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right)$

$\frac{dy}{dx}\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$

$\frac{dy}{dx} = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \frac{xy\ln y - y^2}{xy\ln x - x^2}$,provided x and y are not both equal to e.

14. A soft drink company wants to make a new drink and sell it in a cylindrical shaped can. The can must hold 12π cubic inches of soda. The construction of this can costs the company \$0.50 per square inch to make the top of the can, \$1.50 per square inch to make the bottom of the can, and \$1.00 per square inch to make the sides of the can. Use calculus to determine the radius and height of the can which would minimize the company's cost to make the can.



We are given that this can must hold 12π cubic inches of soda. Hence $\pi r^2 h = 12\pi$. Thus $h = \frac{12}{r^2}$. We want to minimize the cost of the can. The top and bottom of the can are both circles, hence the area of the top and bottom of the can are both πr^2. The side of the can has area $2\pi rh$. Thus, considering the cost given in the statement of this problem,

$C = 0.5(\pi r^2) + 1.50(\pi r^2) + 1.00(2\pi rh) = 2\pi r^2 + 2\pi rh$. Substituting $h = \frac{12}{r^2}$ in for h yields

$C = 2\pi r^2 + \frac{24\pi}{r}$, where $r > 0$.

$C' = 4\pi r - \frac{24\pi}{r^2} = \frac{4\pi r^3 - 24\pi}{r^2}$

$= \frac{4\pi(r^3 - 6)}{r^2}$. This yields a critical number of $r = \sqrt[3]{6}$. Now, $C'' = 4\pi + \frac{48\pi}{r^3} > 0$, hence C is concave up at this critical number, yielding a minimum. Thus the radius of the can is $r = \sqrt[3]{6}$ inches and the height is $h = \frac{12}{(\sqrt[3]{6})^2}$ inches.

15. A tank contains 1000 L of sugar water with 8 kg of dissolved sugar. Pure water enters the tank at a rate of 4 liters per minute. The solution is kept mixed and exits the tank at the same rate. How much sugar (in kg) is in the tank after 7 minutes?

Let $y = y(t)$ be the amount of sugar in the tank. Then $y(0) = 8$. Since pure water is going in the tank,

$\frac{dy}{dt} = \left(0\frac{kg}{L}\right)\left(4\frac{L}{min}\right) - \left(\frac{y}{1000}\frac{kg}{L}\right)\left(4\frac{L}{min}\right)$

Hence $\frac{dy}{dt} = -\frac{1}{250}y\frac{kg}{min}$.

Now, we recall that if $\frac{dy}{dt} = ky$, then $y(t) = y_0 e^{kt}$. Thus since we have $\frac{dy}{dt} = -\frac{1}{250}y$, $y(t) = y_0 e^{kt} = 8e^{(-1/250)t}$. Thus after 7 minutes,

$y(7) = 8e^{(-7/250)}$ kg of sugar.

16. Let C denote the graph of $y = \arcsin x$, $-1 \leq x \leq 1$. Determine the point(s) on C where the tangent line to C is parallel to the line $y = 2x$.

We first need to solve $\frac{dy}{dx} = 2$ for x.

$\frac{1}{\sqrt{1 - x^2}} = 2$. Thus $1 = 2\sqrt{1 - x^2}$. Squaring both sides of this equation yields

$1 = 4(1 - x^2)$, hence $x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$. Now, $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ and $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$. Thus the points are

$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{\pi}{3}\right)$.