Exam II Solutions (Form A)

1. D \( \frac{d}{dx}(2 \sin x - \cos x) = 2 \cos x - (- \sin x) = 2 \cos x + \sin x \)

2. E \( y' = \sec x \tan x - 1 \); when \( x = 0 \), \( y' = \sec 0 \tan 0 - 1 = -1 \)

3. C \( f'(x) = 8(2x - 1)^7(2) \) using the Chain Rule, so \( f'(0) = 8(-1)^7(2) = -16 \)

4. B The derivatives of \( \sin x \) follow the pattern \( \cos x, -\sin x, -\cos x, \sin x, \) ... so the tenth derivative is the same as the second derivative, \( -\sin x \).

5. C \( r'(t) = <2e^{2t}, 1/3 t^{-2/3}> \) so \( r'(1) = <2e^2, 1/3> \)

6. E Newton’s Method says that \( x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^3 - 2 \cdot 2 - 5}{3(2^2) - 2} = 2 - \frac{1}{10} = 21/10 \)

7. B Begin with the exponent: \( \lim_{x \to -\infty} \frac{1}{x} = -\infty \). Thus \( \lim_{x \to -\infty} \frac{2}{1 + 3^{1/x}} = \frac{2}{1 + 0} = 2 \)

8. B The graphs of the functions are shown below. Using the horizontal line test, \( f_1 \) and \( f_3 \) are NOT one-to-one.

9. D Either use the formula \( L(x) = f(a) + f'(a)(x - a) \) or just find the equation of the tangent line. \( f(\pi) = e^{\sin \pi} = e^0 = 1 \). \( f'(x) = e^{\sin x} \cos x \), so \( f'(\pi) = e^{\sin \pi} \cos \pi = -1 \). The equation is \( L(x) = 1 + (-1)(x - \pi) = \pi + 1 - x \).

10. A Switch \( x \) and \( y \) and solve for \( y \): \( x = \frac{1}{2 + y^2}, 2x + xy^2 = 1, xy^2 = 1 - 2x, y = \sqrt{\frac{1 - 2x}{x}} \) (we take the positive square root since the domain of the original function is \([0, \infty)\)). The domain of this function is when the fraction is positive, which occurs when \( x \in (0, 1/2) \).

11. A The limit is nothing more than the definition of \( f'(a) \), where \( f(x) = \tan^2 x \) and \( a = \frac{\pi}{4} \). \( f'(x) = 2 \tan x \sec^2 x \), so \( f'(\pi/4) = 2 \tan \left(\frac{\pi}{4}\right) \sec^2 \left(\frac{\pi}{4}\right) = 2(1)(\sqrt{2})^2 = 4 \).

12. Using the Product Rule, \( y' = 2xe^x + x^2e^x = (2x + x^2)e^x, y'' = (2 + 2x)e^x + (2x + x^2)e^x = (2 + 4x + x^2)e^x \), \( y''' = (4 + 2x)e^x + (2 + 4x + x^2)e^x = (6 + 6x + x^2)e^x \).

Substituting all derivatives into the differential equation yields \( (6 + 6x + x^2)e^x - 3(2 + 4x + x^2)e^x = 0 \)

13. \( y' \sqrt{x - 1} + (y(1/2)(x - 1)^{-1/2}) + y' = 2xy' + 2y \). Moving the terms with \( y' \) to the left and terms without \( y' \) to the right yields \( y'(x - 1)^{1/2} + (1/2)x(y - 1)^{-1/2}y' - 2xy = 2y - (1/2)y(x - 1)^{-1/2} - (y - 1)^{1/2} \), so \( y' = \frac{2y - (1/2)y(x - 1)^{-1/2} - (y - 1)^{1/2}}{-2x + (1/2)x(y - 1)^{-1/2} + (x - 1)^{1/2}} \).
14. i) The relating formula is \( V = x^3 \), given \( \frac{dV}{dt} = -2 \). At the instant when \( V = 27 \), we know \( x = 3 \).

Differentiate and substitute: \( \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \), \( -2 = 3(3)^2 \frac{dx}{dt} \), or \( \frac{dx}{dt} = -\frac{2}{27} \text{ min} \).

ii) The relating formula is \( S = 6x^2 \). Differentiate: \( \frac{dS}{dt} = 12x \frac{dx}{dt} \). At the instant when \( x = 3 \), \( \frac{dx}{dt} = -\frac{2}{27} \), so substitute: \( \frac{dS}{dt} = 12(3) \left( -\frac{2}{27} \right) = -\frac{8}{3} \text{ m}^2 \text{ min}^{-1} \).

15. i) Using the quotient rule \( x'(t) = \frac{(1 + t)(-1) - (1 - t)(1)}{(1 + t)^2} = \frac{-2}{(1 + t)^2} \).

ii) Using the quotient rule \( y'(t) = \frac{(1 + t)(1/2)t^{-1/2} - (t^{1/2} - 1)(1)}{(1 + t)^2} \).

iii) The slope is found by \( \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \). When \( x = 0 \) and \( y = 0 \), we have \( t = 1 \), so the slope is

\[
\left. \frac{dy}{dx} \right|_{t=1} = \frac{(1+1)(1/2)(1)^{-1/2} - (1^{1/2} - 1)(1)}{(1+1)^2} = -\frac{1}{2}
\]

16. The slope of the tangent line is given by \( g'(1) = \frac{1}{f'(g(1))} \). \( g'(1) = \frac{1}{f'(0)} = \frac{1}{e^{0(3 \cdot 0 + 2)}} = \frac{1}{2} \). To find \( g(1) \), solve \( f(x) = 1 \). \( e^{x^3+2x} = 1 \) when \( x^3 + 2x = x(x^2 + 2) = 0 \), or \( x = 0 \) (since \( x^2 + 2 \neq 0 \)).

So \( g'(1) = \frac{1}{f'(0)} = \frac{1}{e^{0(3 \cdot 0 + 2)}} = \frac{1}{2} \). Since \( g(1) = 0 \), the tangent line passes through the point \((1, 0)\). Therefore, the equation is \( y - 0 = \frac{1}{2}(x - 1) \), or \( y = \frac{1}{2}x - \frac{1}{2} \).