Solutions to Exam III Form A

1. **D** By the definition of logarithmic functions, $x + 2 = 2^3 = 8$, so $x = 6$.

2. **E** The domain is all $x$ such that $-1 \leq x - 2 \leq 1$, which means $1 \leq x \leq 3$.

3. **C** The derivative is $y' = 1 - \frac{1}{1 + x^2}$, so the slope is $y'(1) = 1 - \frac{1}{1 + 1^2} = \frac{1}{2}$.

4. **C** Use logarithmic differentiation:

$$\frac{dy}{y} = \ln(x^{\tan x} - x \ln x)$$

$$\frac{y'}{y} = \frac{1}{x} + \sec^2 x \ln x$$

$$y' = x^{\tan x} \left( \frac{\tan x}{x} + \ln x \sec^2 x \right)$$

5. **E** As $x \to 0^+$, $\ln x \to -\infty$ and $x^2 \to 0$, so the limit is either $\infty$ or $-\infty$. Since the numerator is negative (approaching $-\infty$) and the denominator is positive, the limit is $-\infty$. (NOTE that since the limit is in a determinate form, we should NOT use L'Hospital’s Rule)

6. **D** The limit of the indeterminate form $\frac{0}{0}$, so we use L'Hospital’s Rule this time:

$$\lim_{x \to 0} \frac{\sin x + x \cos x}{-\sin x}$$

This is still of the form $\frac{0}{0}$, so we can either use L'Hospital’s Rule again, or note that the expression simplifies to

$$\lim_{x \to 0} -1 - \frac{x}{\sin x} \cos x = -1 - (1)(0) = -1 - 1 = -2$$

7. **C** Assuming exponential growth, the equation is $y = Ce^{kt}$. When $t = 0$, $y = 500$, so $500 = Ce^{k(0)}$, or $C = 500$. When $t = 3$, $y = 8000$, so $8000 = 500e^{3k}$. Solving for $k$ yields $k = \frac{1}{3} \ln 16$.

(NOTE: you can also solve using $e^k = 16^{1/3}$. Therefore, the expression is $y = 500e^{(1/3 \ln 16)t} = 500e^{(\ln 16^{1/3})t} = 500(16^{1/3})^t$.

8. **B** Find the second derivative and equate to 0:

$$y' = \frac{x}{2} + \cos x, y'' = \frac{1}{2} - \sin x = 0$$

$$\sin x = \frac{1}{2}, \text{ or } x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6}$$

(since the interval is $0 \leq x \leq \pi$). Testing $f''$ at a value in each of the subintervals yields $f'' > 0$ when $0 \leq x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x \leq \pi$ and $f'' < 0$ when $\frac{\pi}{6} < x < \frac{5\pi}{6}$.

Therefore, there are points of inflection when $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

9. **E** Let $y = \sin^{-1} t$. Then $\sin y = t = \frac{t}{1}$, which can be illustrated using the right triangle below (since $t > 0$). Therefore, $\tan y = \frac{t}{\sqrt{1 - t^2}}$.

10. **D** Since $f$ only has one critical number at $x = 1$, $f'(1) = 0$ (since $f$ is differentiable everywhere) and $f'(x) \neq 0$ for all other values of $x$. To find the critical numbers of $h$, set $h'(x) = 2xf'(x^2) = 0$. This is true when $x = 0$ and when $x^2 = 1$, or $x = \pm 1$.

11. **D** Dividing by $x^2$ yields $g'(x) = 2 + \frac{3}{x}$, or $g(x) = 2x + 3 \ln x + C$. Since $g(e) = 1$, $1 = 2e + 3 \ln e + C$, or $C = -2 - 2e$. Therefore, $g(x) = 2x + 3 \ln x - (2 + 2e)$.
12. 

(a) The domain of \( H \) is all \( x \) such that \( \sin^{-1} x \) is defined (meaning \( x > 0 \)) and \( -1 \leq x \leq 1 \). Both are true when \( 0 < x \leq 1 \).

(b) Using the Chain Rule, \( H'(x) = \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} \).

(c) The domain of the derivative must be a subset of \((0, 1]\) (the domain of the original function). The only value in this interval where \( f'(x) \) is not defined is \( x = 1 \). Therefore, the domain of the derivative is \( 0 < x < 1 \).

13. 

(a) Using the product rule, \( f'(x) = e^{-2x^2} + xe^{-2x^2}(-4x) = e^{-2x^2}(1 - 4x^2) \). Differentiate this using the product rule again yields \( f''(x) = e^{-2x^2}(-4x)(1 - 4x^2) + e^{-2x^2}(-8x) = -4xe^{-2x^2}(1 - 4x^2 + 2) = -4xe^{-2x^2}(3 - 4x^2) \).

(b) The critical numbers occur when \( f' = 0 \). Since \( e^{-2x^2} \neq 0 \) for all \( x \), the solution is \( 1 - 4x^2 = 0 \), or \( x = \pm \frac{1}{2} \).

(c) The critical numbers divide the \( x \)-axis into three subintervals. Test a value from each subinterval by substituting into \( f' \). \( f'(x) < 0 \) when \( x < -\frac{1}{2} \) and when \( x > \frac{1}{2} \), and \( f'(x) > 0 \) when \(-\frac{1}{2} < x < \frac{1}{2} \). Therefore, \( f \) is increasing on \( \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right) \) and \( f \) is decreasing on \( \left(-\frac{1}{2}, \frac{1}{2}\right) \).

(d) From part (c), the local maximum is \( f \left( \frac{1}{2} \right) = \frac{1}{2} e^{-1/2} \).

(e) \( f \left( \frac{-1}{2} \right) = -\frac{1}{2} e^{-1/2} \). Since \( f(x) \to 0 \) as \( x \to \pm \infty \), the absolute maximum is \( \frac{1}{2} e^{-1/2} \) and the absolute minimum is \( f \left( \frac{1}{2} \right) = -\frac{1}{2} e^{-1/2} \).

(f) \( f''(x) = 0 \) when \( x = 0 \), \( x = \pm \frac{\sqrt{3}}{2} \). Test a value from each of the four subintervals by substituting into \( f'' \). \( f''(x) < 0 \) when \( x < -\frac{\sqrt{3}}{2} \) and when \( 0 < x < \frac{\sqrt{3}}{2} \). \( f''(x) > 0 \) when \( -\frac{\sqrt{3}}{2} < x < 0 \) and when \( x > \frac{\sqrt{3}}{2} \). Therefore, \( f \) is concave up on \( \left(-\frac{\sqrt{3}}{2}, 0\right) \cup \left(\frac{\sqrt{3}}{2}, \infty\right) \).

(g) The absolute maximum of \( f \) is \( \frac{1}{2} e^{-1/2} = \frac{1}{2} e^{-1/2} \), which is greater than \( \frac{1}{4} \). Therefore, from the graph of \( f \) shown below (determined by information in the previous parts), there are two values of \( x \) where \( f(x) = \frac{1}{4} \).
14. Let \( y = \lim_{x \to \infty} \left( \frac{x - 3}{x + 4} \right)^x \). Then \( \ln y = \lim_{x \to \infty} x \ln \left( \frac{x - 3}{x + 4} \right) = \lim_{x \to \infty} \frac{\ln \left( \frac{x - 3}{x + 4} \right)}{\frac{1}{x}} \). Applying L’Hospital’s Rule yields \( \lim_{x \to \infty} \left( \frac{x - 3}{x + 4} \right)^x = \lim_{x \to \infty} \frac{x + 4}{x - 3} \cdot \frac{7}{x(x + 4)^2} \cdot (-x^2) = (1)(-7) = -7 \). Since \( \ln y \to -7 \), \( y \to e^{-7} \).

15. Our goal is to maximize \( S = 2\pi r^2 + 2\pi rh \) with the condition that \( V = \pi r^2 h = 22 \). Then \( h = \frac{22}{\pi r^2} \). Substitution into \( S \) yields \( S = 2\pi r^2 + 2\pi r \left( \frac{22}{\pi r^2} \right) = 2\pi r^2 + \frac{44}{r} \). Differentiate to find the critical values: \( S' = 4\pi r - \frac{44}{r^2} = 0 \), \( 4\pi r^3 - 44 = 0 \), \( r = \sqrt[3]{\frac{11}{\pi}} \). Show this critical value is a minimum by showing \( S'' = 4\pi \left( \frac{88}{r^3} \right) > 0 \). The dimensions of the can are \( r = \sqrt[3]{\frac{11}{\pi}} \) inches and \( h = \frac{22}{\pi \left( \frac{11}{\pi} \right)^{2/3} } = 2 \sqrt[3]{\frac{11}{\pi}} \) inches.