Solutions to Exam III Form B

1. A By the definition of logarithmic functions, \( x - 2 = 2^3 = 8 \), so \( x = 10 \).

2. B The domain is all \( x \) such that \( -1 \leq x + 2 \leq 1 \), which means \( -3 \leq x \leq -1 \).

3. D The derivative is \( y' = \frac{1}{1 + x^2} - 1 \), so the slope is \( y'(1) = \frac{1}{1 + 1^2} - 1 = -\frac{1}{2} \).

4. B Use logarithmic differentiation:
   \[
   \ln y = \ln(x^\sec x) = \sec x \ln x \\
   \frac{y'}{y} = \sec x \left( \frac{1}{x} \right) + \sec x \tan x \ln x \\
   y' = x^\sec x \left( \frac{\sec x}{x} + \ln x \sec x \tan x \right)
   \]

5. B As \( x \to 0^+ \), \( \ln x \to -\infty \) and \( x^2 \to 0 \), so the limit is either \( \infty \) or \( -\infty \). Since the numerator is negative (approaching \( -\infty \)) and the denominator is positive, the limit is \( -\infty \). (NOTE that since the limit is in a determinate form, we should NOT use L'Hospital's Rule)

6. C The limit is of the indeterminate form \( \frac{0}{0} \), so we do use L'Hospital's Rule this time:
   \[
   \lim_{x \to 0} \frac{\sin x + x \cos x}{\sin x} \text{. This is still of the form } \frac{0}{0} \text{, so we can either use L'Hospital's Rule again, or note that the expression simplifies to } \\
   \lim_{x \to 0} 1 + \frac{x}{\sin x} (\cos x) = 1 + 1(\cos 0) = 1 + 1 = 2.
   \]

7. D Assuming exponential growth, the equation is \( y = Ce^{kt} \). When \( t = 0 \), \( y = 500 \), so \( 500 = Ce^{k(0)} \), or \( C = 500 \). When \( t = 3 \), \( y = 9000 \), so \( 9000 = 500e^{3k} \). Solving for \( k \) yields \( k = \frac{1}{3} \ln 18 \). (NOTE: you can also solve using \( e^k = 18^{1/3} \)). Therefore, the expression is \( y = 500e^{(1/3 \ln 18)t} = 500e^{\ln(18^{1/3})} = 500(18)^{t/3} \).

8. C Find the second derivative and equate to 0:
   \[
   y' = \frac{\sqrt{3}x}{2} + \cos x, \ y'' = \frac{\sqrt{3}}{2} - \sin x = 0
   \]
   \[
   \sin x = \frac{\sqrt{3}}{2}, \text{ or } x = \frac{\pi}{3} \text{ and } x = \frac{2\pi}{3} \text{ (since the interval is } 0 \leq x \leq \pi) \text{. Testing } f'' \text{ at a value in each of the subintervals yields } f'' > 0 \text{ when } 0 \leq x < \frac{\pi}{3} \text{ or } \frac{2\pi}{3} < x \leq \pi \text{ and } f'' < 0 \text{ when } \frac{\pi}{3} < x < \frac{2\pi}{3} \text{. Therefore, there are points of inflection when } x = \frac{\pi}{3} \text{ and } x = \frac{2\pi}{3}.
   \]

9. B Let \( y = \cos^{-1} t \). Then \( \cos y = t = \frac{t}{\sqrt{1 - t^2}} \), which can be illustrated using the right triangle below (since \( t > 0 \)). Therefore, \( \tan y = \frac{\sqrt{1 - t^2}}{t} \).

10. A Since \( f \) only has one critical number at \( x = 1 \), \( f'(1) = 0 \) (since \( f \) is differentiable everywhere) and \( f'(x) \neq 0 \) for all other values of \( x \). To find the critical numbers of \( h \), set \( h'(x) = 2xf'(x^2) = 0 \). This is true when \( x = 0 \) and when \( x^2 = 1 \), or \( x = \pm 1 \).

11. B Dividing by \( x^2 \) yields \( g'(x) = 3 + \frac{2}{x} \). So \( g(x) = 3x + 2 \ln x + C \). Since \( g(e) = 1, 1 = 3e + 2 \ln e + C \), or \( C = -1 - 3e \). Therefore, \( g(x) = 2x + 3 \ln x - (1 + 3e) \).

12. .
(a) The domain of $H$ is all $x$ such that $\sin^{-1} x > 0$ (meaning $x > 0$) and $-1 \leq x \leq 1$. Both are true when $0 < x \leq 1$.

(b) Using the Chain Rule, $H'(x) = \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1 - x^2}}$.

(c) The domain of the derivative must be a subset of $(0, 1]$ (the domain of the original function). The only value in this interval where $f'(x)$ is not defined is $x = 1$. Therefore, the domain of the derivative is $0 < x < 1$.

13. .

(a) Using the product rule, $f'(x) = e^{-2x^2} + xe^{-2x^2}(-4x) = e^{-2x^2}(1 - 4x^2)$. Differentiate this using the product rule again yields $f''(x) = e^{-2x^2}(-4x)(1 - 4x^2) + e^{-2x^2}(-8x) = -4xe^{-2x^2}(1 - 4x^2 + 2) = -4xe^{-2x^2}(3 - 4x^2)$

(b) The critical numbers occur when $f' = 0$. Since $e^{-2x^2} \neq 0$ for all $x$, the solution is $1 - 4x^2 = 0$, or $x = \pm \frac{1}{2}$.

(c) The critical numbers divide the $x$-axis into three subintervals. Test a value from each subinterval by substituting into $f'$. $f'(x) < 0$ when $x < -\frac{1}{2}$ and when $x > \frac{1}{2}$, and $f'(x) > 0$ when $-\frac{1}{2} < x < \frac{1}{2}$. Therefore, $f$ is increasing on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $f$ is decreasing on $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.

(d) From part (c), the local minimum is $f\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-1/2}$.

(e) $f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1/2}$. Since $f(x) \to 0$ as $x \to \pm \infty$, the absolute maximum is $\frac{1}{2}e^{-1/2}$ and the absolute minimum is $f\left(\frac{1}{2}\right) = -\frac{1}{2}e^{-1/2}$.

(f) $f''(x) = 0$ when $x = 0$, $x = \pm \frac{\sqrt{3}}{2}$. Test a value from each of the four subintervals by substituting into $f''$. $f''(x) < 0$ when $x < -\frac{\sqrt{3}}{2}$ and when $0 < x < \frac{\sqrt{3}}{2}$. $f''(x) > 0$ when $-\frac{\sqrt{3}}{2} < x < 0$ and when $x > \frac{\sqrt{3}}{2}$. Therefore, $f$ is concave down on $\left(-\infty, -\frac{\sqrt{3}}{2}\right) \cup \left(0, \frac{\sqrt{3}}{2}\right)$.

(g) The absolute maximum of $f$ is $\frac{1}{2}e^{-1/2} = \frac{1}{2\sqrt{e}}$, which is greater than $\frac{1}{4}$. Therefore, from the graph of $f$ shown below (determined by information in the previous parts), there are two values of $x$ where $f(x) = \frac{1}{4}$.
14. Let \( y = \lim_{x \to \infty} \left( \frac{x - 3}{x + 5} \right)^x \). Then \( \ln y = \lim_{x \to \infty} x \ln \left( \frac{x - 3}{x + 5} \right) = \lim_{x \to \infty} \frac{\ln \left( \frac{x - 3}{x + 5} \right)}{\frac{1}{x}} \). Applying L’Hospital’s Rule yields \( \lim_{x \to \infty} \frac{(x+5)x-(x-3)1}{(x+5)^2} = \lim_{x \to \infty} \frac{x+5}{x-3} \cdot \frac{8}{(x+5)^2} \cdot (-x^2) = (1)(-8) = -8 \). Since \( \ln y \to -8 \), \( y \to e^{-8} \).

15. Our goal is to maximize \( S = 2\pi r^2 + 2\pi rh \) with the condition that \( V = \pi r^2 h = 24 \). Then \( h = \frac{24}{\pi r^2} \). Substitution into \( S \) yields \( S = 2\pi r^2 + 2\pi r \left( \frac{24}{\pi r^2} \right) = 2\pi r^2 + \frac{48}{r} \). Differentiate to find the critical values: \( S' = 4\pi r - \frac{48}{r^2} = 0 \), \( 4\pi r^3 - 48 = 0 \), \( r = \sqrt[3]{\frac{12}{\pi}} \). Show this critical value is a minimum by showing \( S'' = 4\pi + \frac{96}{r^3} > 0 \). The dimensions of the can are \( r = \sqrt[3]{\frac{12}{\pi}} \) inches and \( h = \frac{24}{\pi (\frac{12}{\pi})^{2/3}} = 2 \sqrt[3]{\frac{12}{\pi}} \) inches.