GUIDELINES

• In Part 1 (Problems 1–13), mark your responses on your ScanTron form using a No: 2 pencil. *For your own record, mark your choices on the examination paper as well.* ScanTrons will be collected at the conclusion of the examination; they will *not* be returned.

• In Part 2 (Problems 14–17), present your solutions in the space provided. *Show all your work* neatly and concisely, and *indicate your final answer clearly.* You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

• Be sure to write your name, section number, and version letter of the examination on the ScanTron form.

• Calculators *should not be used* throughout the examination. Mobile phones *must be switched off*. Examinees found using a mobile phone during the course of the test are subject to being ejected from the examination hall, and to being given a zero on the examination.
PART 1 (52 points)
Each question is worth 4 points. Mark your responses on the ScanTron form and on the examination paper itself.

1. What is the value of \( \arccos(\sin 2\pi) \)?
   (a) \(-\pi/2\)
   (b) \(2\pi\)
   (c) \(\pi/2\)
   (d) 0
   (e) \(\pi\)

2. What is the value of \( \cot(\arcsin(3/4)) \)?
   (a) \(3/5\)
   (b) \(3/\sqrt{7}\)
   (c) \(\sqrt{7}/3\)
   (d) \(5/3\)
   (e) \(7/3\)

3. Determine all possible values of the real number \( x \) such that \( \log_2(x^2 + 1) \geq 1 \).
   (a) \((\infty, -1] \cup [1, \infty)\)
   (b) \([1, \infty)\)
   (c) \([-1, 1]\)
   (d) \((\infty, 0) \cup (0, \infty)\)
   (e) \((\infty, \infty)\)
4. Let \( f(x) = \arccos(\sqrt{x}) \), for \( 0 < x < 1 \). Compute \( f'(x) \).

(a) \( -\frac{1}{\sqrt{1-x}} \)

(b) \( -\frac{1}{2\sqrt{x} - x^2} \)

(c) \( -\frac{1}{2\sqrt{x} - x^3} \)

(d) \( -\frac{1}{\sqrt{x} - x^2} \)

(e) \( \frac{\sec(\sqrt{x}) \tan(\sqrt{x})}{2\sqrt{x}} \)

5. Let \( f(x) = x^{\cos x} \), for \( x > 0 \). Compute the slope of the tangent to the graph of \( f \) at the point \( (\pi/2, f(\pi/2)) \).

(a) 0

(b) \( \frac{\pi}{2} \ln \left( \frac{2}{\pi} \right) \)

(c) \( \frac{\pi}{2} \ln \left( \frac{\pi}{2} \right) \)

(d) \( \ln \left( \frac{\pi}{2} \right) \)

(e) \( \ln \left( \frac{2}{\pi} \right) \)

6. The growth of a bacterial culture is modelled by the equation \( y(t) = y_0 e^{kt} \), where \( y(t) \) denotes the bacteria count after \( t \) hours, \( y_0 \) denotes the initial population of the culture, and \( k \) is a constant. Given that the bacterial count was 400 after 2 hours and 6400 after 6 hours, determine the value of \( k \).

(a) \( \ln(16) \)

(b) 4

(c) 16

(d) \( \ln(16)/4 \)

(e) \( \ln 4 \)
7. Calculate \( \lim_{x \to 0} 2^{\frac{x^2}{\sin x - x}} \).
   (a) \(2^{-6}\)
   (b) \(-2^6\)
   (c) \(2^6\)
   (d) \(2^{1/6}\)
   (e) \(2^{-1/6}\)

8. Compute \( \lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}} \).
   (a) 1
   (b) 0
   (c) -2
   (d) \(\infty\)
   (e) \(-\infty\)

9. Let \( f \) be a function whose derivative is given by \( f'(x) = e^{-2x^2} - 4x^2 e^{-2x^2} \), for \(-\infty < x < \infty\). Find the interval(s) in which \( f \) is increasing.
   (a) \((-\infty, 1/2)\)
   (b) \((-1/2, 1/2)\)
   (c) \((1/2, \infty)\)
   (d) \((-\infty, 0)\)
   (e) \((-\infty, -1/2) \cup (1/2, \infty)\)
10. Suppose that the second derivative of a function $f$ is given by $f''(x) = \frac{x^2(x^2 - 1)(x - 1)}{x^2 + 1}$, for $-\infty < x < \infty$. Determine the value(s) of $x$ for which there is an inflection point on the graph of $f$.

(a) only 0 and 1

(b) only $-1$

(c) only $-1$ and 1

(d) only $-1$, 0, and 1

(e) the graph has no points of inflection.

11. Find the function $F$, defined on $(0, \infty)$, and satisfying the following pair of conditions:

$F'(x) = \frac{2x^3 - x^2 + 3}{x^2}$ for $0 < x < \infty$, and $F(1) = 1$.

(a) $2x^2 - x - \frac{3}{x} + 3$

(b) $x^2 - x - \frac{6}{x^3} + 7$

(c) $2 - \frac{6}{x^3} + 5$

(d) $x^2 - x - \frac{3}{x} + 4$

(e) $x^2 - \frac{3}{x} + 3$

12. Suppose that $f$ is a function defined on $(0, \infty)$, and that its derivative is given by $f'(x) = \frac{1 - \ln x}{x^2}$, for $0 < x < \infty$. Determine the interval in which the graph of $f$ is concave upwards.

(a) $(e^{3/2}, \infty)$

(b) $(0, e^{3/2})$

(c) $(0, 3/2)$

(d) $(3/2, \infty)$

(e) $(0, e)$
13. Let \( f(x) = x + 2 \cos x \), for \( \pi \leq x \leq 2\pi \). Determine the absolute maximum value of \( f \) in the interval \([\pi, 2\pi]\).

(a) \( \pi - 2 \)

(b) \( 2(\pi + 1) \)

(c) \( \pi/6 + \sqrt{3} \)

(d) \( 7\pi/6 + \sqrt{3} \)

(e) the function does not achieve an absolute maximum in the said interval.

**PART 2 (48 points)**

*Present your solutions to the following problems (14–17) in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.*

14. (9 points) Calculate

\[
\lim_{x \to \infty} (3x + 1)^{\frac{2}{3 \ln(4x^2 + 5)}}.
\]

\[
\frac{2}{3 \ln(4x^2 + 5)}
\]

\[
\frac{2 \ln (3x+1)}{3 \ln (4x^2+5)}
\]

Now

\[
\lim_{x \to \infty} \frac{2 \ln (3x+1)}{3 \ln (4x^2+5)} = \left[ \frac{\infty}{\infty} \right]
\]

\[
= \lim_{x \to \infty} \frac{2}{3x+1} \cdot \frac{3}{3 \ln (4x^2+5)}
\]

\[
= \lim_{x \to \infty} \frac{6 (4x^2+5)}{24 x (3x+1)}
\]

\[
= \frac{24}{24} = \frac{1}{3}
\]

so the required limit is

\[ e^{\frac{1}{3}}, \text{ via } \bigcirc. \]
15. Consider the functions $F$ and $G$ defined as follows:

\[ F(x) = \arctan(e^{2x}) + e^{\arctan(2x)}; \quad G(x) = \arcsin(\ln x). \]

(i) (2 points) Determine the domain of $F$.

\((-\infty, \infty), \text{ because the arctan and exponential functions are well defined throughout}\)

\((-\infty, \infty).\)

(ii) (4 points) Determine the domain of $G$.

\[-1 \leq \ln(x) \leq 1 \quad \Rightarrow \quad e^{-1} \leq x \leq e, \text{ so the domain of } G \text{ is } [e^{-1}, e].\]

(iii) (6 points) Differentiate $F$ with respect to $x$.

\[
F'(x) = \frac{2e^{2x}}{1 + e^{2x}} + \frac{2e^{\arctan(2x)}}{1 + 4x^2}, \quad -\infty < x < \infty.
\]

(iv) (3 points) Differentiate $G$ with respect to $x$.

\[
G'(x) = \frac{1}{\sqrt{1 - (\ln x)^2}} \cdot \frac{1}{x}, \quad \text{for } e^{-1} < x < e.
\]
16. (9 points) The top and bottom margins of a poster are each 6 cm, and the side margins are each 4 cm. If the area of the printed material on the poster is fixed at 384 square centimeters, use calculus methods to find the dimensions of the poster with the smallest area. **Draw a figure and label all variables.** Your analysis should also include showing that the answer does, in fact, yield the said minimum value.

\[
\begin{align*}
\text{Given:} \quad & (u-8)(v-12) = 384 \\
\text{Problem: Maximize} \quad & A = uv \\
& = u \left[ \frac{384}{u-8} + 12 \right] \\
& \text{Subject to} \quad u > 8, \\
\text{Now} \quad & A'(u) = \frac{384}{u-8} + 12 - \frac{384u}{(u-8)^2} \\
& = \frac{384(u-8) + 12(u-8)^2 - 384u}{(u-8)^2} \\
& = \frac{12(u-8)^2 - 8 \cdot 384}{(u-8)^2}, \quad u > 8. \\
\end{align*}
\]

As \( A'(u) \) exists for every \( u > 8 \), the only possible critical points for \( A \) must come from solving the equation \( A'(u) = 0 \) for \( u > 8 \). Now \( A'(u) = 0 \) \( \iff \) \( 12 \left[ (u-8)^2 - 256 \right] = 0 \) \( \Rightarrow \) \( u = 24 \). The accompanying first derivative analysis reveals that \( A \) achieves a local and absolute minimum at \( u = 24 \). Thus the required dimensions are

\[
\begin{align*}
& u = 24 \text{ cm and } v = 36 \text{ cm.}
\end{align*}
\]
17. Let

\[ f(x) = \frac{x^3}{3} - 4|x| + 6, \quad \text{for } -\infty < x < \infty. \]

(i) (7 points) Find all the critical numbers of \( f \).

\( f \) is nondifferentiable at \( x = 0 \) [because the function is not differentiable at \( x = 0 \)], whilst the function \( x - 4|x| \) is not differentiable at \( x = 0 \). So \( x = 0 \) is a critical number.

Now \( f(x) = \begin{cases} 
\frac{x^3}{3} + 4x + 6, & \text{if } x \leq 0 \\
\frac{x^3}{3} - 4x + 6, & \text{if } x > 0 
\end{cases} \)

So \( f'(x) = \begin{cases} 
x^2 + 4, & \text{if } x \leq 0 \\
x^2 - 4, & \text{if } x > 0 
\end{cases} \)

As \( f'(x) > 0 \) for \( x < 0 \), and \( f'(x) = 0 \) for \( x > 0 \), it implies that \( x^2 - 4 = 0 \), or \( x = \pm2 \). Hence, \( x = \pm2 \) is also a critical point.

\( f \) has two critical points: \( 0 \) and \( 2 \).

(ii) (6 points) Determine the intervals in which \( f \) increases, and those in which it decreases.

\[ \begin{array}{c|c|c|c}
\text{Interval} & f' & f & \text{Behavior} \\
\hline
(-\infty, 0) & - & \uparrow & \downarrow \\
(0, 2) & + & \downarrow & \uparrow \\
(2, \infty) & - & \uparrow & \downarrow \\
\end{array} \]

\( f \) increases on \( (-\infty, 0] \) and also on \([2, \infty)\).

It decreases on \([0, 2]\).

(iii) (2 points) Determine all the values of \( x \) which give rise to local maxima of \( f \), as well as those which yield local minima.

\( x = 0 \) produces a local max, and \( x = 2 \) yields a local min.
LAST NAME (print): ______________________________________

FIRST NAME (print): ______________________________________

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