Spring 2013 Math 151

Exam III Version B Solutions

1. E Switch $x$ and $y$ and solve for $y$: $x = \frac{8y - 1}{2y + 5}$.
   
   
   $$2xy + 5x = 8y - 1, \quad 2xy - 8y = -5x - 1,$$
   
   $$y = \frac{5x + 1}{2x - 8} = \frac{5x + 1}{8 - 2x}.$$

2. A Using properties of logarithms, 

   $$(e^{3\ln x})(\ln(e^{2x})) = e^{\ln(x^3)}(\ln(e^{2x})) = (x^3)(2x) = 2x^4.$$

3. E Using properties of logarithms, $f(x) = \ln(x^4) + \ln(e^x) = 4\ln x + x$, so $f'(x) = \frac{4}{x} + 1$.

4. E Let $y = \arcsin\left(\frac{1}{4}\right)$. Using the reference triangle below, $\tan y = \frac{1}{\sqrt{15}}$.

5. C The limit is of the indeterminate form $\frac{0}{0}$, so apply L’Hospital’s Rule:

   $$\lim_{x \to 0} \frac{\sin x}{x^2 - x^2} = \lim_{x \to 0} \frac{\cos x}{-2x} = -\frac{1}{-2\pi} = \frac{1}{2\pi}.$$

6. C $f$ is increasing when $f'$ is positive, and $f$ is concave down when $f''$ is decreasing. Therefore, the correct interval (positive and decreasing) is (3, 5).

7. A $f$ has a critical value when $f'(x) = \cos x = 0$, or $x = -\frac{\pi}{2}$ (on the given interval). Evaluate $f$ at this critical value and at the endpoints: 

   $$f(-\pi) = \sin(-\pi) = 0, \quad f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1, \quad \text{and} \quad f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$ 

   Therefore, the absolute maximum on the given interval is $\frac{1}{2}$.

8. E $f'(x) = x^2(x - 2)(x - 1) = 0$ when $x = 0$, $x = 1$, and $x = 2$. Testing the signs on each subinterval, we find that $f' > 0$ on $(-\infty, 0) \cup (0, 1) \cup (2, \infty)$ and $f' < 0$ on $(1, 2)$. Therefore, $f$ has a relative maximum at $x = 1$ only.

9. A $g'(5) = \frac{1}{f'(g(5))}$. Since $g = f^{-1}$, if $y = g(5)$, then $f(y) = 5$, which means $y = 2$. Therefore, $g'(5) = \frac{1}{f'(2)} = -2$.

10. D We cannot simply cancel since $\frac{4\pi}{3}$ is outside the range of $f(x) = \arccos x$. $\cos \frac{4\pi}{3} = -\frac{1}{2}$, and $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

11. D The function $f$ is decreasing and concave down. Therefore, $f' < 0$ and $f'' < 0$.

12. D $f'(x) = \frac{1}{3} - \frac{2}{3}x^{-1/3}$, which is zero when $x^{-1/3} = \frac{1}{2}$, or $x^{1/3} = 2$, or $x = 8$. However, $f'$ does not exist when $x = 0$. Therefore, the critical values are $x = 0$ and $x = 8$.

13. C $f''(x) = 4x^3 - 4x^2 = 4x^2(x - 1) = 0$ when $x = 0$ and $x = 1$. Testing the signs on each interval, we find that $f'' > 0$ on $(1, \infty)$ and $f'' < 0$ on $(-\infty, 0) \cup (0, 1)$. Therefore, $f$ has an inflection point at $x = 1$ only.

14. B Differentiate to find which function yields $xe^x$ as the answer. Since $\frac{d}{dx}((x - 1)e^x = (x - 1)(e^x) + e^{-x}(1) = xe^x - e^x + e^x = xe^x$, the antiderivative is $(x - 1)e^x + C$.

15. E The statement in the problem means that $f'(x) = 2x + 3$, so $f(x) = x^2 + 3x + C$. Substituting $x = 1$ and $f(x) = 2$ yields $2 = 1^2 + 3(1) + C$, or $C = -2$. Therefore, $f(x) = x^2 + 3x - 2$. 

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16. \( a(t) = \mathbf{v}'(t) = (1 + 3 \sin t, \ 2 - \cos t) \), so \( \mathbf{v}(t) = (t - 3 \cos t + C_x, \ 2t - \sin t + C_y) \). Since \( \mathbf{v}(0) = (5, 4) = (0 - 3 + C_x, \ 0 - 0 + C_y) \), we have \( C_x = 8 \) and \( C_y = 4 \). Therefore, \( \mathbf{v}(t) = (t - 3 \cos t + 8, \ 2t - \sin t + 4) \).

17. 

(a) Use logarithmic differentiation: Let \( y = (1 - 3x)^{1/x} \). Then \( \ln y = \frac{1}{x} \ln(1 - 3x) \) and \( \frac{dy}{dx} = x \left( \frac{\frac{1}{1-3x} - \ln(1 - 3x)}{x^2} \right) \). Therefore, 

\[
\frac{dy}{dx} = y \left( \frac{\frac{1}{1-3x} - \ln(1 - 3x)}{x^2} \right), \quad \text{or} \quad \frac{dy}{dx} = (1 - 3x)^{1/x} \frac{\frac{1}{1-3x} - \ln(1 - 3x)}{x^2}.
\]

(b) \( \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 - 3x)}{x} \). Since this limit is of the form \( \frac{0}{0} \), apply L'Hôpital's Rule: \( \lim_{x \to 0} \frac{-3/x}{1} = -3 \). Since \( \lim_{x \to 0} y = -3, \lim_{x \to 0} (1 - 3x)^{1/x} = \exp(-3) \).

18. Using properties of logarithms, we have \( \log_4(3x + x^2) = 1 \). Rewriting as an exponential (or applying base-4 exponential to each side) yields \( 3x + x^2 = 4^1 = 4, \) or \( x^2 + 3x - 4 = 0, \) or \( (x + 4)(x - 1) = 0. \) Therefore, our solutions are \( x = 1 \) or \( x = -4. \) However, we must substitute each of these back into the original equation to determine if they are in the domain of the expression on the left. If \( x = 1 \) we have \( \log_4(1) + \log_4(-4), \) which is undefined over the real numbers. If \( x = -4, \) we have \( \log_4(4) + \log(1), \) which is defined and is, in fact, equal to \( 1. \) Therefore, the solution is \( x = -4 \) only.

19. 

(a) Let \( y \) be the temperature of the metal ball. The problem can be modeled by the differential equation \( \frac{dy}{dt} = -\frac{1}{3} (y - 21) \) (negative since the temperature is decreasing). The solution to this differential equation is \( y - 21 = Ce^{(-1/3)t}, \) or \( y = 21 + Ce^{(-1/3)t}. \) When \( t = 0, \) \( y = 36, \) so we have \( 36 = 21 + Ce^{-1/3(0)}, \) or \( C = 15. \) Therefore, the temperature of the ball at any time \( t \) is \( y = 21 + 15e^{-1/3t}. \) Hence, after 2 minutes, the temperature is \( y = 21 + 15e^{-2/3 \cdot 2} \cdot C \).

(b) Using the information in part (a), we seek to solve \( 25 = 21 + 15e^{-1/3t}, \) or \( 4 = 15e^{-1/3t}, \) or \( \frac{4}{15} = e^{-1/3t}. \) Apply the natural logarithm function to both sides: \( \ln \left( \frac{4}{15} \right) = -\frac{1}{3} t, \) or \( t = -3 \ln \left( \frac{4}{15} \right) \) minutes.

20. Choose a point \( (x, y) \) on the ellipse to be the corner of the rectangle opposite the origin. Our goal is to maximize \( A = xy, \) the area of the rectangle, under the constraint that \( x^2 + \frac{y^2}{9} = 1. \) The location of the maximum of \( A \) will be the same as the location of the maximum of \( A = A^2 = x^2y^2. \) Solving our constraint for \( y \) yields \( y^2 = 9 - x^2, \) so our goal is to maximize \( A = x^2(9 - x^2) = 9x^2 - x^4, \) with \( 0 \leq x \leq 1. \) Differentiate to find the critical value(s): \( 18x - 36x^3 = 0, \) \( 18x(1 - 2x^2) = 0, \) so \( x = 0 \) or \( x = \pm \sqrt{\frac{1}{2}}. \) The only critical value in the interior of the region is \( x = \sqrt{\frac{1}{2}}. \) Show this is indeed a maximum by either: 1) evaluating \( A \) at the interior critical value and both endpoints, 2) testing the value of \( A' \) on each subinterval, or 3) using the Second Derivative Test. Therefore, the base of the largest rectangle is \( \sqrt{\frac{1}{2}}. \) (no units specified).

21. When \( x = \frac{1}{2}, \ y = \arctan(-3^{1/2}) = \arctan(-\sqrt{3}) = -\frac{\pi}{3}. \) Use the derivative to find the slope: \( f'(x) = \frac{1}{1 + (-3x)(\ln 3)}, \) so \( m = f' \left( \frac{1}{2} \right) = -\frac{\sqrt{3} \ln 3}{1 + 3} = -\frac{\sqrt{3} \ln 3}{4}. \) Therefore, the equation of the tangent line is \( y + \frac{\pi}{3} = -\frac{\sqrt{3} \ln 3}{4} \left( x - \frac{1}{2} \right). \)