13. (B) Since $f'$ changes sign from $-\to+\text{ as }x$ increases through 2, we conclude that $f$ has a local minimum at $x = 2$ by the First Derivative Test (henceforth: FDT).

14. (B) Note that $f'$ changes sign from $-\to+\text{ only at }x = 2$, at which $f$ has a local minimum by the FDT.

15. With $f(x) = (x - 2)e^{3x}$, we examine signs of $f'(x) = (3x - 5)e^{3x}$ and $f''(x) = (9x - 12)e^{3x}$.
   - $f$ is increasing for $x > \frac{5}{3}$, decreasing for $x < \frac{5}{3}$.
   - $f$ has a local minimum at $x = \frac{5}{3}$ by the FDT.
   - $f$ is concave up for $x > \frac{5}{3}$, concave down for $x < \frac{5}{3}$.
   - $f$ has an inflection point at $x = \frac{5}{3}$: $f''$ changes sign.

16. The area is $A = 2xy = 2x\sqrt{25 - x^2}$, $0 \leq x \leq 5$ (to allow for degenerate rectangles in order to apply the CIM). Solve $A'(x) = 2\sqrt{25 - x^2} - \frac{2x^2}{\sqrt{25 - x^2}} = 0$ to get $x = \frac{5}{2}\sqrt{2} \approx 3.54 \in [0,5]$. Since $A$ is zero for $x = 0,5$ and positive otherwise, maximal area occurs for width $2x = 5\sqrt{2}$ and height $y = \sqrt{25 - x^2} = \frac{5}{2}\sqrt{2}$.

17. Now $y = (1 + \frac{2}{3})^{3x}$ is an indeterminate power $1^\infty$ as $x \to \infty$. Let $\ln y = 3x\ln(1 + \frac{2}{3}) = 3x(1 + 2x^{-1})/x^{-1}$, an indeterminate quotient $0/0$. Use l’Hospital’s Rule: $\lim_{x \to \infty} \ln y = \lim_{x \to \infty} -6x^2/(1 + 2x^{-1}) = \lim_{x \to \infty} \frac{6}{1 + \frac{2}{3}} = 6$. Then $\lim_{x \to \infty} y = \lim_{x \to \infty} e^{\ln y} = e^6$.

18. Take the acceleration $r''(t) = [e^t, \cos t - 1]$ and antidifferentiate twice, resolving constants as you go.

# Exam 3A: Solutions

## Wed, 29/Apr 2015 Art Belmonte

1. (B) Since $f'$ changes sign from $-\to+\text{ as }x$ increases through 2, we conclude that $f$ has a local minimum at $x = 2$ by the First Derivative Test (henceforth: FDT).

2. (E) As $x \downarrow 3$, we have $\ln(\frac{x-3}{x}) \to \ln 0^+ = \ln 0^+ = -\infty$.

3. (A) Use logarithmic differentiation: $\ln f = \pi x$, $x = \frac{3}{2}$, $x = \frac{2\pi}{3}$.

4. (D) Let $\ln f$ since $\ln f \to 0$.

5. (B) $f(x) = 3\sin^{-1}x + \frac{3x}{\sqrt{1-x^2}} - \frac{3}{\sqrt{3}}$.

6. (A) Via properties of logarithms, the expression is equal to $\ln \left(\frac{(x^2+1)^3}{x^2+1}\right)$ or $\ln \left(\frac{x^2+1}{x}\right)^3$.

7. (C) As $x \to 0$, we obtain $0/0$. Use l’Hospital’s Rule:

$$\lim_{x \to 0} \tan^{-1}2x = \lim_{x \to 0} \frac{2x}{3x} = \frac{2}{3}.$$

8. (A) Use Closed Interval Method (henceforth CIM).

9. (A) Use logarithmic differentiation: $\ln y = (\ln x)^2$, whence $\frac{1}{y} \frac{dy}{dx} = \frac{2\ln x}{x}$. Thus $\frac{dy}{dx} = x^{\ln x} \left(\frac{2\ln x}{x}\right)$.

10. (B) Let $\theta = \tan^{-1}x$. Then $\tan \theta = \frac{x}{\sqrt{1-x^2}}$. Therefore see $\theta = \sqrt{x^2+1}$ as seen in the diagram below.

11. (B) We have $\cos^{-1}(\cos \frac{2\pi}{3}) = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$.

12. (E) Interior terms in this telescoping sum collapse, leaving $2^1 - 2^{101} = 2 - 2^{101}$.

13. (B) For $f(x) = \ln(x^2) = 2\ln x$, we have $f(e) = 2$ and $f''(e) = \frac{2}{e}$. The tangent line is $y - 2 = \frac{2}{e}(x - e)$.

14. (E) Combine logarithms, then exponentiate to obtain $x^2 - 4 = 0$, whence $x = \pm 2$. Now $x = 2$ is the only solution: $x = -2$ doesn’t satisfy the original equation since $\ln(-2)$ is undefined.
1. (A) Use Closed Interval Method (henceforth CIM). With \( f(x) = 3x - x^3 \), we have \( f'(x) = 3 - 3x^2 = 0 \) when \( x = \pm1 \). Note that \( -1 \notin [0,3] \); toss it out! Now \( f((0,1,3]) = (0,2,-18) \). The absolute maximum is \( f(1) = 2 \); the absolute minimum is \( f(3) = -18 \).

2. (B) We have \( \cos^{-1}(\cos \frac{\pi}{3}) = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3} \).

3. (A) Use logarithmic differentiation: \( \ln y = (\ln x)^2 \), whence \( \frac{1}{y} \frac{dy}{dx} = \frac{2\ln x}{x} \). Thus \( \frac{dy}{dx} = y\ln x (\frac{2\ln x}{x}) \).

4. (E) As \( x \downarrow 3 \), we have \( \ln (\frac{x-3}{x}) \rightarrow \ln 0^+ = -\infty \).

5. (D) Let \( y \) be the number of bacteria. Then \( y = 125e^{x^2} \).
Solve \( 350 = 125e^{2x} \) to obtain \( k = \frac{1}{2} \ln \frac{14}{5} \), whence \( y = 125 \left( \frac{14}{5} \right)^{\frac{x^2}{2}} \). After 24 hours, \( y = 125 \left( \frac{14}{5} \right)^{12} \).

6. (E) Interior terms in this telescoping sum collapse, leaving \( 2^1 - 2^{101} = 2 - 2^{101} \).

7. (B) Since \( f' \) changes sign from \(-\) to \(+\) as \( x \) increases through \( 2 \), we conclude that \( f \) has a local minimum at \( x = 2 \) by the First Derivative Test (henceforth: FDT).

8. (B) At \( x = -\frac{1}{2}, f'(x) = 3 \sin^{-1}x + \frac{3x}{\sqrt{1-x^2}} = -\frac{\pi}{2} - \frac{3}{\sqrt{3}} \).

9. (A) Now \( 0 = f'(x) = (2 \cos x - 1) \sin x \) on \( 0 < x < 2\pi \) for \( x = \pi, x = \frac{\pi}{2}, x = \frac{5\pi}{2} \).

10. (B) Let \( \theta = \tan^{-1}x \). Then \( \tan \theta = \frac{x}{\sqrt{x^2 + 1}} \) as seen in the diagram below.

11. (B) For \( f(x) = \ln (\sqrt{x^2 + 1}) \), we have \( f'(x) = 2 \) and \( f''(x) = \frac{2}{x^2} \). The tangent line is \( y - 2 = \frac{2}{x} (x - e) \).

12. (A) Via properties of logarithms, the expression is equal to \( \ln \left( \frac{(x^2+3)^3}{x(x^2+1)^{\frac{1}{2}}} \right) \) or \( \ln \left( \frac{(x^2+3)^3}{x^{\frac{1}{2}}(x^2+1)} \right) \).

13. (D) Note that \( f' \) changes sign from \(-\) to \(+\) only at \( x = 2 \), at which \( f \) has a local minimum by the FDT.

14. (E) Combine logarithms, then exponentiate to obtain \( x^2 - 4 = 0 \), whence \( x = \pm2 \). Now \( x = 2 \) is the only solution: \( x = -2 \) doesn’t satisfy the original equation since \( \ln(-2) \) is undefined.

15. (C) As \( x \rightarrow 0 \), we obtain \( 0/0 \). Use l’Hospital’s Rule:
\[
\lim_{x \rightarrow 0} \frac{\tan^{-1}2x}{3x} = \lim_{x \rightarrow 0} \frac{\frac{2}{1+(2x)^2}}{3} = \frac{2}{3}.
\]

16. With \( f(x) = (x-1)e^{x^2}, \) we examine signs of \( f'(x) = (2x-1)e^{x^2} \) and \( f''(x) = 4xe^{x^2} \).
- \( f \) is increasing for \( x > \frac{1}{2} \), decreasing for \( x < \frac{1}{2} \).
- \( f \) has a local minimum at \( x = \frac{1}{2} \) by the FDT.
- \( f \) is concave up for \( x > 0 \), concave down for \( x < 0 \).
- \( f \) has an inflection point at \( x = 0 \): \( f'' \) changes sign.

17. The area is \( A = 2xy = 2\sqrt{16 - x^2}, \) \( 0 \leq x \leq 4 \) (to allow for degenerate rectangles in order to apply the CIM). Solve \( (x') = 2\sqrt{16 - x^2} - \frac{2e^{\sqrt{16-x^2}}}{0} = 0 \) to get \( x = 2\sqrt{2} \approx 2.83 \in [0,4] \). Since \( A \) is zero for \( x = 0 \) and positive otherwise, maximal area occurs for width \( 2x = 4\sqrt{2} \) and height \( y = \sqrt{16 - x^2} = 2\sqrt{2} \).

18. Now \( y = (1 + \frac{1}{x})^x \) is an indeterminate power \(^{a^\infty}\) as \( x \rightarrow \infty \). Let \( \ln y = 5x\ln (1 + \frac{1}{2}) = 5\ln (1 + 2x)/x \), an indeterminate quotient \( 0/0 \). Use l’Hospital’s Rule:
\[
\lim_{x \rightarrow \infty} \frac{\ln y}{\ln} = \lim_{x \rightarrow \infty} \frac{-10x^{-2}/(1 + 2x^{-1})}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{10}{1 + \ln x} = 10.
\]
Then \( \lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} e^{\ln y} = e^{10} \).

19. Take the acceleration \( r''(t) = [e^t + t, \cos t - 1] \) and antiderivative twice, resolving constants as you go.
\[
r'(t) = \left[ e^t + \frac{1}{2}t^2, \sin t - t \right] + C
\]
\[
[-1,2] = r'(0) = [1,0] + C
[2,4] = C
\[
r'(t) = \left[ e^t + \frac{1}{2}t^2 - 2, \sin t - t \right]
\[
r(t) = \left[ e^t + \frac{1}{6}t^3 - 2t^2 - \cos t - \frac{1}{2}t^2 + 2t \right] + K
\]
\[
[4,-12] = r(0) = [1,1] + K
[3,-11] = K
\[
r'(t) = \left[ e^t + \frac{1}{6}t^3 - 2t^2 - 3, -\cos t - \frac{1}{2}t^2 + 2t - 11 \right]
\]

20. \( \) Let \( y = 7^\tan x \). Then \( \ln y = x \tan x \cdot \ln 7 \), whence \( \frac{1}{y} \frac{dy}{dx} = (1 \tan x + x \sec^2 x) \ln 7 \). Therefore, we have \( \frac{dy}{dx} = 7^\tan x (\tan x + x \sec^2 x) \ln 7 \).
- The Chain Rule gives \( \frac{d}{dx} (\sin^{-1} (x^3)) = \frac{3x^2}{\sqrt{1-(x^3)^2}} \).
- Via Chain Rule, \( \frac{d}{dx} (\log_4 (\ln x)) = \frac{1}{x \ln 4 \ln 4} \).