Fall 2015  Math 151  
Exam 2H  Solutions  
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1. So \( f(x) = \frac{x}{1+x^2} \) & \( f'(x) = \frac{-1-x^2}{(1+x^2)^2} \). The tangent is  
\[ y = f(3) + f'(3)(x-3) = \frac{3}{10} - \frac{2}{25} (x-3) = \frac{27}{50} - \frac{2}{25} x. \]

2. Now \( p = fg \) and \( q = f/g \) imply \( p' = f'g + fg' \) and \( q' = \frac{ef-gf'}{x^2} \). Via table, \( p'(4) = 24 \) and \( q'(4) = \frac{36}{25} \).

3. For \( t = 20 \) (year 1920), backward and forward slopes are \( 1750-1860 = 11 \) and \( 2070-1860 = 21 \); average slope: \( \frac{11+21}{2} = 16 \). Similarly, for 1980 average of slopes is \( \frac{80+29}{2} = 79.5 \). Rates are in millions of people per year.

4. Now \( s(t) = t^3 - 6t^2 + 9t \) and \( v(t) = 3t^2 - 12t + 9 \) or \( v(t) = 3(t-1)(t-3) \).

(a) Velocity is 0 for \( t = 1 \) or \( t = 3 \), positive on \( [0,1) \cup (3,5] \), and negative on \( (1,3) \).

(b) In the first 5 seconds, the distance traveled is \( |s(1) - s(0)| + |s(3) - s(1)| + |s(5) - s(3)| \) or 28 m. Or integrate speed: \( f_0^5 |v(t)| \, dt = 28 \).

5. With \( y = 2 \sin x \), we have \( y' = 2 \sin x + 2 \cos x \). At \( x = \frac{\pi}{2} \), the tangent line is \( y = \pi + 2 \left( \frac{\pi}{2} - x \right) \) or \( y = 2x \).

6. Recall \( f(x) = x/\sqrt{1-x^2} \).

(a) Graph at end shows jump discontinuity at \( x = 0 \).

(b) As \( x \to 0^- \), \( f(x) \to -\frac{\sqrt{2}}{2} \), but as \( x \to 0^+ \), \( f(x) \to \frac{\sqrt{2}}{2} \), verifying this jump discontinuity.

7. Now \( h(x) = f(g(x)) \), and \( h'(x) = f'(g(x))g'(x) \). So \( h'(1) = f'(g(1))g'(1) = f'(2)(g'(1) = (5)(6) = 30 \) via table. Similarly, \( k'(1) = 36 \) for \( k(x) = g(f(x)) \).

8. With \( f(x) = \sqrt{x^2-x+1} \), \( f'(x) = \frac{x}{\sqrt{x^2-x+1}} \).

for \( x = \pm \frac{3^{1/4}}{3} = \pm \frac{1}{\sqrt{3}} \pm 0.76 \).

9. Recall the hyperbola’s equation, \( x^2 - 4xy + y^2 = 4 \).

(a) When \( x = 0 \), \( y = \pm 2 \).

(b) Via implicit differentiation, \( y' = \frac{-2y}{x} \).

At \( (0, \pm 2) \), the tangent lines have slope 2.

10. [See page 2 for solution to #10 (hard).]

11. With position \( \mathbf{r}(t) = [-\frac{1}{2}t^2, t] \), we have velocity \( \mathbf{v}(t) = [-t, 1] \) and acceleration \( \mathbf{a}(t) = [-1, 0] \). Thus \( \mathbf{v}(1) = [-1, 1] \) and \( \mathbf{a}(1) = [-1, 0] \). Graph at end.

12. With position \( \mathbf{r}(t) = [e^t \cos t, e^t \sin t] \), we have velocity \( \mathbf{v}(t) = [e^t (\cos t - \sin t), e^t(\cos t + \sin t)] \). At \( t = 0 \), the velocity is \( [1, 1] \) and speed is \( \sqrt{2} \).

13. With \( x = 3 \cos t \) and \( y = -2 \sin t \), the trig identity \( \cos^2 t + \sin^2 t = 1 \) implies \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \), an ellipse traversed clockwise for increasing \( t \). Graph at end.

14. The derivatives of \( f(x) = xe^x \) are \( (x+1)e^x \), \( (x+2)e^x \), \( (x+3)e^x \), \( (x+4)e^x \) ... So \( f^{(n)}(x) = (x+n)e^x \).

15. Now \( \mathbf{r}(t) = [x(t), y(t)] = [\cos t, \sin t] \) has period \( 2\pi \).

- **Horizontal tangents:** solve \( \frac{dx}{dt} = -\sin t = 0 \); get \( t = \frac{\pi}{2}, \frac{3\pi}{2} \in [0, 2\pi] \); note \( \frac{dy}{dt} = -\cos t \neq 0 \) thereat; points \( (x, y) \) are \((1, e) \) and \((1, -e) \).

- **Vertical tangents:** solve \( \frac{dy}{dt} = -\cos t = 0 \); get \( t = 0, \pi \in [0, 2\pi] \); note \( \frac{dx}{dt} = e^t \sin t \neq 0 \) thereat; points \( (x, y) \) are \((e, 1) \) and \((-e, 1) \).

16. Recall \( x = t^3 - 3t \) and \( y = t^2 + t + 1 \). The horizontal tangent occurs when \( \frac{dy}{dt} = 2t + 1 = 0 \); i.e., at \( t = -\frac{1}{2} \).

Note \( \frac{dx}{dt} = 3t^2 - 3 \neq 0 \) thereat. The lowest point is \( (x, y) = (\frac{11}{8}, \frac{3}{4}) \) \((1.375, 0.75) \).

17. Volume of water is \( V = \pi x^2 h = 25\pi h \). Therefore, \( \frac{dV}{dh} = 25\pi x^2 \) or \( \frac{dh}{dt} = \frac{dy}{dt} \approx 0.0382 \) m/min.

18. So \( \cos \theta = \frac{\frac{x}{2}}{\sqrt{\frac{49}{3}} \sqrt{\frac{49}{16}}} \) gives \( -\sin \theta \) \( \frac{d\theta}{dt} = \frac{dx}{dt} \frac{dy}{dt} = \frac{-dy}{dx} \frac{dy}{dt} \).

At stated instant, \( x = 3, y = 4 \), hyp = 5, \( \frac{dy}{dx} = 1 \). Thus \( \frac{d\theta}{dt} = -\frac{1}{5(4/5)} = -\frac{1}{4} \) rad/s \( \approx -14.32^\circ /s \).

19. Let \( x \) be edge length of cube. Volume is \( V = x^3 \) and surface area is \( S = 6x^2 \). Maximum error estimates in \( V \) and \( S \) are \( dV = 3x^2 dx \) & \( ds = 12dx \) (the geometrically obvious approximations). At stated instant, \( x = 30 \), \( dx = 0.1 \) cm \( \approx 0.001 \) cm & \( ds = 36 \) cm & 36 cm respectively.

20. Recall \( g(2) = 4 \) and \( g'(x) = (x^2 + 5)^{1/2} \) for all \( x \).

(a) \( L(x) = g(2) + g'(2) \cdot (x-2) = -4 + 3(x-2) \).

(b) Now \( g''(x) = \frac{x}{\sqrt{x^2 + 5}} \).

Quadratic approximation is \( Q(x) = g(2) + g'(2) \cdot (x-2) + \frac{1}{2}g''(2) \cdot (x-2)^2 \) or \( Q(x) = -4 + 3(x-2) + \frac{1}{2}(x-2)^2 \).

21. At desired point, \( y \)-values and slopes are equal. Solve \( x^3 - 3x + 4 = 3(x^3 - x) \) and \( 3x^2 - 3 = 3(2x - 1) \) to obtain \( (x, y) = (2, 6) \). Common tangent is \( y = 9x - 12 \).

22. [See page 2 for solution to #22 (hard).]

23. Solve \( y = f(x) = 2 + \sqrt{4+7x} \) for \( x \) to obtain \( x = \frac{(y-2)^2 - 4}{7} \), whence \( f^{-1}(x) = \frac{(x-2)^2 - 4}{7} \). See reverse.

24. With \( f(x) = x + e^x \), \( f(0) = 1 \). So \( g(1) = f^{-1}(1) = 0 \) and \( g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{(1+e)} |_{a=0} = \frac{1}{2} \).


#10 Recall the graph of the ellipse and its two tangent lines through \( P(12, 3) \).

![Graph of the ellipse and tangent lines](image)

- For the ellipse \( x^2 + 4y^2 = 36 \), we have \( y' = \frac{3x}{4y} \) via implicit differentiation. The slope of the tangent line to the ellipse at \((x_0, y_0)\) is \( m = \frac{-x_0}{4y_0} \), our first equation.

- The point \((x_0, y_0)\) is on the ellipse. This gives our second equation, \( x_0^2 + 4y_0^2 = 36 \).

- The point \((x_0, y_0)\) and \( P(12, 3) \) are on the tangent line at \((x_0, y_0)\). This gives \( 3 - y_0 = m(12 - x_0) \), which is our third equation.

- Solving these three equations simultaneously for \( m \), \( x_0 \), and \( y_0 \) gives two solutions. The first is \( m = 0 \), \( x_0 = 0 \), \( y_0 = 3 \), which you can immediately see by looking at the graph. The second is \( m = \frac{2}{3} \), \( x_0 = \frac{24}{5} \), and \( y_0 = -\frac{9}{5} \).

- With this information, the two tangent lines are \( y = 3 \) and \( y = -\frac{9}{5} + \frac{2}{3} (x - \frac{24}{5}) \) or \( y = \frac{2}{3}x - 5 \).

#22 Recall the graph of the parabola \( y = x^2 \) and the inscribed circle \((x - h)^2 + (y - k)^2 = 1^2\).

- The circle’s center \((h, k) = (0, k)\) has \( x \)-coordinate 0 due to the symmetry of the parabola \( y = x^2 \) w.r.t. the \( y \)-axis. Let \( P(c, c^2) \) be where the circle and parabola intersect in the right half-plane (so \( c > 0 \)).

- Here’s the essence. \( P \) lies on both curves and slopes of the tangent lines to said curves at \( P \) are the same. This gives two equations (below) and two unknowns, \( k \) and \( c > 0 \) for which we must simultaneously solve.

- Since \( h = 0 \), the circle’s equation is \( x^2 + (y - k)^2 = 1 \). Implicit differentiation gives \( y' = \frac{x}{k - y} = \frac{c}{c^2 - k} \) at \( P \) for the slope of the tangent line to the circle. The slope of the parabola at \( x = c \) is \( 2c \). Setting the slopes equal gives our first equation, \( \frac{c}{c^2 - k} = 2c \). Next, \( P \) lies on the circle. So our second equation is \( c^2 + (c^2 - k)^2 = 1 \).

- Solving the two equations simultaneously for \( c \) and \( k \) with \( c > 0 \) yields \( c = \frac{\sqrt{2}}{2} \) and \( k = \frac{5}{4} \). So the center of the circle is \((h, k) = (0, \frac{5}{4})\).

- As the B52s said, “The party’s gone out of bounds!”

#23 With \( f(x) = 2 + \sqrt{4 + 7x} \), we saw \( f^{-1}(x) = \frac{(x - 2)^2 - 4}{7} \).

The domain of \( f \) is \([-\frac{4}{7}, \infty)\). Its range is \([2, \infty)\). Hence the domain of \( f^{-1} \) is the range of \( f \), \([2, \infty)\), and the range of \( f^{-1} \) is the domain of \( f \), \([-\frac{4}{7}, \infty)\)—next time.

**Graphs**

![Graph of Problem 6](image)

![Graph of Problem 11](image)

![Graph of Problem 13](image)