SOLUTIONS

MATH 151, SPRING 2021
COMMON EXAM 2 - ONLINE EXAM VERSION A

The work out problems make up 44 points of the exam, while the multiple choice problems make up 56 points (3.5 points each) for a total of 100 points. **No calculator is allowed!**

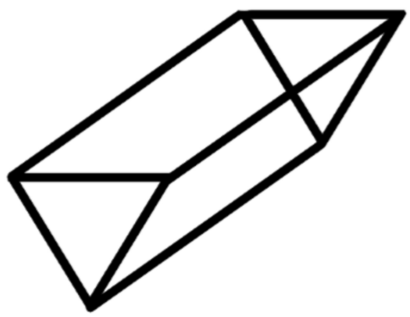
PART I: WORK OUT PROBLEMS

Directions: Present each of your solutions on an empty sheet/side of paper. *Show all of your work* neatly and concisely and *box your final answer.* You will be graded not merely on the final answer, but also the quality and correctness of the work leading up to it.

1. (8 pts) Find the values of a and b so that the line $5x + 2y = a$ is tangent to the function $y = bx\sqrt{x}$ when $x = 9$.

2. (8 pts) Find $\frac{dy}{dx}$ for the equation: $\tan(7x^2) = 3^{2y} + y^5 e^{6x}$.

3. (9 pts) A trough is 18 m long and its ends have the shape of isosceles triangles that are 8 m across the top and have a height of 3 m. Water is being drained from it at a rate of 12 m^3/min. Find the rate at which the height of the water in the tank is changing when the height of the water is 1 m.



For the remaining work out problems, find the derivative, but do not simplify during or after taking the derivative. Your final answer should also not include y and should only be in terms of x.

4. (6 pts) Find $\frac{dy}{dx}$ for $y = \dfrac{\ln(\pi) - x^7}{\sec(2x)\ln(3x)}$.

5. (5 pts) Find $\frac{dy}{dx}$ for $y = \arctan(x^2 e^{4x})$.

6. (8 pts) Find $\frac{dy}{dx}$ for $y = (4 - 3x)^{\sin(5x)}$.

Multiple choice problems begin on the next page...

1. (8 pts) Find the values of a and b so that the line $5x + 2y = a$ is tangent to the function $y = bx\sqrt{x}$ when $x = 9$.

The tangent line has slope $y'\big|_{x=9}$ so need y':

$$y = bx\sqrt{x} = bx^{3/2} \Rightarrow y' = \frac{3}{2}bx^{1/2}$$

Evaluating at $x=9$ gives the slope of the tangent line is $\frac{3}{2}b(9)^{1/2} = \frac{9}{2}b$

But the slope of $5x+2y=a$ is $-5/2$ $\Rightarrow$ $\frac{9}{2}b = -5/2$ $\Rightarrow$ $\boxed{b = \frac{-5}{9}}$

The must also go through the same pt at $x=9$ (that is, have the same y-value)

tangent line: $5(9) + 2y = a \Rightarrow y = \frac{a-45}{2}$

function: $y = b \cdot 9\sqrt{9} = -\frac{5}{9} \cdot 9 \cdot 3 = -15$

Setting equal: $\frac{a-45}{2} = -15 \Rightarrow a = -30 + 45 \Rightarrow \boxed{a = 15}$

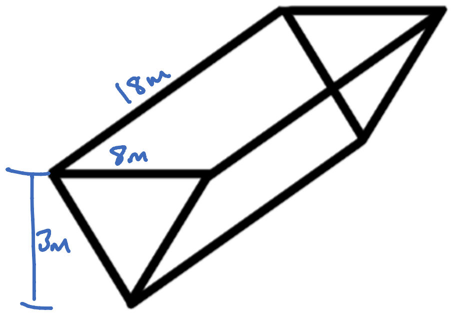
2. (8 pts) Find $\frac{dy}{dx}$ for the equation: $\tan(7x^2) = 3^{2y} + y^5e^{6x}$.

$$\frac{d}{dx}\left(\tan(7x^2) = 3^{2y} + y^5 \cdot e^{6x}\right)$$

$$\Rightarrow \left(\sec^2(7x^2)\right)(14x) = 3^{2y}\ln(3) \cdot 2\frac{dy}{dx} + 5y^4\frac{dy}{dx}e^{6x} + y^5 \cdot 6e^{6x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{14x\sec^2(7x^2) - 6y^5e^{6x}}{2 \cdot 3^{2y}\ln(3) + 5y^4e^{6x}}}$$

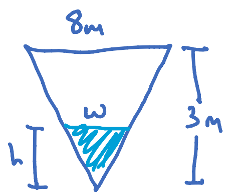
3. (9 pts) A trough is 18 m long and its ends have the shape of isosceles triangles that are 8 m across the top and have a height of 3 m. Water is being drained from it at a rate of 12 m^3/min. Find the rate at which the height of the water in the tank is changing when the height of the water is 1 m.



Set h = height of the water at time t

w = width of the water at time t

V = volume of water at time t



Know: $V = \frac{1}{2} wh(18) = 9\,wh$

$\frac{dV}{dt} = -12\ m^3/min$ (negative since draining)

Want: $\frac{dh}{dt}$ when $h = 1$ m

Using similar triangles: $\frac{h}{3} = \frac{w}{8} \Rightarrow w = \frac{8h}{3} \Rightarrow V = 9\left(\frac{8h}{3}\right)h = 24h^2$

Relate the rates by implicitly diff: $\frac{d}{dt}\left(V = 24h^2\right) \Rightarrow \frac{dV}{dt} = 48h \cdot \frac{dh}{dt}$

Plug in what's known & solve: $-12 = 48(1) \cdot \frac{dh}{dt} \Rightarrow \boxed{\frac{dh}{dt} = -\frac{1}{4}\ m/min}$

(the height is decreasing at 1/4 m/min)

For the remaining work out problems, find the derivative, but do not simplify during or after taking the derivative. Your final answer should also not include y and should only be in terms of x.

4. (6 pts) Find $\dfrac{dy}{dx}$ for $y = \dfrac{\ln(\pi) - x^7}{\sec(2x)\ln(3x)}$.

5. (5 pts) Find $\dfrac{dy}{dx}$ for $y = \arctan(x^2 e^{4x})$.

6. (8 pts) Find $\dfrac{dy}{dx}$ for $y = (4 - 3x)^{\sin(5x)}$.

#4

$$\boxed{y' = \frac{\left(\sec(2x)\ln(3x)\right)\left(-7x^6\right) - \left(\ln(\pi) - x^7\right)\left[\left(2\sec(2x)\tan(2x)\right)\ln(3x) + \left(\frac{3}{3x}\right)\sec(2x)\right]}{\left(\sec(2x)\ln(3x)\right)^2}}$$

#5

$$\boxed{y' = \frac{1}{1+(x^2e^{4x})^2} \cdot \left(2x e^{4x} + 4e^{4x}x^2\right)}$$

#6 Take ln first: $\ln\left[y = (4-3x)^{\sin(5x)}\right]$

$\Rightarrow \quad \ln(y) = \sin(5x)\ln(4-3x)$

$\frac{d}{dx}$ both sides $\Rightarrow \quad \dfrac{y'}{y} = \left(5\cos(5x)\right)\left(\ln(4-3x)\right) + \left(\dfrac{-3}{4-3x}\right)\left(\sin(5x)\right)$

mult. by y & sub back in $\Rightarrow$

$$\boxed{y' = \left[\left(5\cos(5x)\right)\left(\ln(4-3x)\right) + \left(\frac{-3}{4-3x^2}\right)\left(\sin(5x)\right)\right](4-3x)^{\sin(5x)}}$$

PART II: Multiple Choice. 3.5 points each

Use the table of values given below for differentiable functions f and g to answer Questions #7 and #8.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	-6	12	3	-3
1	1	9	2	6
2	-3	2	1	-2

7. Let $u(x) = f(g(2x))$. Find $u'(1)$.

   (a) -4
   (b) -36 ← correct
   (c) 12
   (d) -18
   (e) -8

$u'(x) = f'(g(2x)) \cdot g'(2x)(2)$

$\Rightarrow u'(1) = f'(g(2)) \cdot g'(2)(2) = f'(1)(-2)(2) = (9)(-4) = \boxed{-36}$

8. Let $v(x) = \dfrac{f(x)}{g(x)}$. Find $v'(-2)$.

   (a) 2 ← correct
   (b) 6
   (c) 5
   (d) -6
   (e) -2

$v'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$\Rightarrow v'(-2) = \dfrac{g(-2)f'(-2) - f(-2)g'(-2)}{(g(-2))^2} = \dfrac{(3)(12) - (-6)(-3)}{(3)^2} = \boxed{2}$

9. Find the t-value(s) so that the curve $x = 2t^3 + 6t^2$, $y = t^3 - 2t$ has a vertical tangent.

   (a) $t = -3, 0$
   (b) $t = -1, 0, 1$
   (c) $t = -1, 1$
   (d) $t = -\sqrt{3}, 0, \sqrt{3}$
   (e) $t = -2, 0$ ← correct

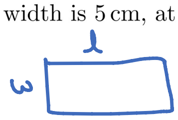
Want $\dfrac{dx}{dy} = \dfrac{6t^2 + 12t}{3t^2 - 2} = 0 \Rightarrow 6t^2 + 12t = 0$

$\Rightarrow 6t(t+2) = 0$

$\Rightarrow \boxed{t = 0, -2}$

10. The length of a rectangle is increasing at a rate of 7 cm/s and its width is decreasing at a rate of 3 cm/s. When the length is 12 cm and the width is 5 cm, at what rate is the area of the rectangle changing at that moment?

   (a) -69 cm/s
   (b) 71 cm/s
   (c) 1 cm/s
   (d) -1 cm/s ← correct
   (e) 69 cm/s



$A = lw$

$\Rightarrow \dfrac{dA}{dt} = l \cdot \dfrac{dw}{dt} + w \cdot \dfrac{dl}{dt}$

$\Rightarrow \left.\dfrac{dA}{dt}\right|_{l=12,\ w=5} = (12)(-3) + (5)(7) = \boxed{-1 \text{ cm/s}}$

11. For what values of x on the interval $[0, 2\pi)$ does the graph of $f(x) = 2\cos(x) + x$ have a horizontal tangent?

(a) $\frac{\pi}{2}, \frac{3\pi}{2}$

(b) $\frac{4\pi}{3}, \frac{5\pi}{3}$

(c) $\frac{\pi}{3}, \frac{2\pi}{3}$

(d) $\frac{7\pi}{6}, \frac{11\pi}{6}$

(e) $\frac{\pi}{6}, \frac{5\pi}{6}$ ← correct

$f'(x) = -2\sin(x) + 1 = 0 \Rightarrow \sin(x) = \frac{1}{2}$

$\Rightarrow \boxed{x = \pi/6,\ 5\pi/6}$

12. Find the equation of the tangent line to the graph of $y^2\cos(x) = 4x + y$ at the point $(0, 1)$.

(a) $y = -3x + 1$

(b) $y = 4x - 4$

(c) $y = 4x + 1$ ← correct

(d) $y = 4x$

(e) $y = -3x$

$\frac{d}{dx}\left(y^2\cos(x) = 4x + y\right)$

$\Rightarrow 2y \cdot \frac{dy}{dx}\cos(x) - \sin(x)\, y^2 = 4 + \frac{dy}{dx}$

eval at $(x,y) = (0,1)$ $\Rightarrow$ $2 \cdot \frac{dy}{dx} = 4 + \frac{dy}{dx} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)} = 4 \Rightarrow \boxed{y = 4x + 1}$

13. Find $f^{(1034)}(x)$, the 1034th derivative of $f(x) = xe^{-x}$.

(a) $f'(x) = (x - 1034)e^{-x}$ ← correct

(b) $f'(x) = 1034x^{-x}$

(c) $f'(x) = (1034 + x)e^{-x}$

(d) $f'(x) = (1034 - x)e^{-x}$

(e) $f'(x) = -1034x^{-x}$

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

$f''(x) = -e^{-x} - (1-x)e^{-x} = (-2+x)e^{-x}$

$f'''(x) = e^{-x} - (-2+x)e^{-x} = (3-x)e^{-x}$

$f^{(4)}(x) = -e^{-x} - (3-x)e^{-x} = (-4+x)e^{-x}$

$\vdots$

$1034/4$ has remainder 2

$\Rightarrow \boxed{(-1034+x)e^{-x}}$

14. The position of a particle is given by the vector function $\mathbf{r}(t) = \langle t^4, te^t \rangle$. Find the acceleration vector of the particle at time $t = 1$.

(a) $\mathbf{a}(t) = \langle 3, 2e \rangle$

(b) $\mathbf{a}(t) = \langle 12, 2e \rangle$

(c) $\mathbf{a}(t) = \langle 4, 3e \rangle$

(d) $\mathbf{a}(t) = \langle 4, 2e \rangle$

(e) $\mathbf{a}(t) = \langle 12, 3e \rangle$ ← correct

$\vec{v}(t) = \vec{r}\,'(t) = \langle 4t^3, e^t + te^t \rangle$

$\Rightarrow \vec{a}(t) = \vec{r}\,''(t) = \langle 12t^2, e^t + (e^t + te^t) \rangle$

$\Rightarrow \vec{a}(1) = \langle 12, e + e + e \rangle = \langle 12, 3e \rangle$

15. At 1:00 PM, a bacteria culture contains 2 million cells and grows at a rate proportional to its size. At 4:00 PM, it has grown to 5 million cells. How many cells will there be in the culture at 7:00 PM, assuming the same rate of growth? All times are on the same day.

(a) 7 million
(b) 8.5 million
(c) 10 million
(d) 12.5 million ← correct
(e) 15 million

$y = y(0)\cdot a^t = (2 \text{ mil})\, a^t$

$5 \text{ mil} = y(3) = (2 \text{ mil})\cdot a^3 \Rightarrow a^3 = 5/2 \Rightarrow a = (5/2)^{1/3}$

$\Rightarrow y = (2 \text{ mil})\left[(5/2)^{1/3}\right]^t \Rightarrow y(6) = (2 \text{ mil})(5/2)^{6/3} = (2 \text{ mil})\left(\frac{25}{4}\right)$

$= \boxed{12.5 \text{ mil}}$

16. Find $f''(x)$ for $f(x) = e^{-5\cos(x)}$.

(a) $5\cos(x)e^{-5\cos(x)} + 25\sin^2(x)e^{-5\cos(x)}$ ← correct
(b) $5\sin(x)e^{-5\cos(x)}$
(c) $\cos(x)e^{-5\cos(x)} + \sin^2(x)e^{-5\cos(x)}$
(d) $e^{5\cos(x)}$
(e) $25\sin^2(x)e^{-5\cos(x)}$

$f'(x) = 5\sin(x)\, e^{-5\cos(x)}$

$\Rightarrow f''(x) = 5\cos(x)\, e^{-5\cos(x)} + \left(5\sin(x) e^{-5\cos(x)}\right)(5\sin(x))$

$\Rightarrow \boxed{f''(x) = 5\cos(x)\, e^{-5\cos(x)} + 25\sin^2(x)\, e^{-5\cos(x)}}$

17. Find the tangent vector of unit length for $\mathbf{r}(t) = \langle t+2, e^{3t}\rangle$ at $t = 0$.

(a) $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$
(b) $\left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$ ← correct
(c) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$
(d) $\langle 1, 3\rangle$
(e) $\langle 2, 1\rangle$

$\vec{r}\,'(t) = \langle 1, 3e^{3t}\rangle \Rightarrow \vec{r}\,'(0) = \langle 1, 3\rangle$

divide by $|\vec{r}\,'(0)| = \sqrt{1^2+3^2} = \sqrt{10}$

$\Rightarrow \boxed{\left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle}$

18. Find the derivative of $f(x) = \ln\left(\dfrac{\sqrt{x^4-6x}}{e^{2x}(x-5)^3}\right)$.

(a) $f'(x) = \dfrac{2x^3-3}{x^4-6x} + \dfrac{3}{x-5} - 2$
(b) $f'(x) = \dfrac{4x^3-6}{x^4-6x} - \dfrac{1}{x-5} - e^{-2x}$
(c) $f'(x) = \dfrac{1}{2(x^4-6x)} - \dfrac{3}{x-5} - 2e^{-2x}$
(d) $f'(x) = \dfrac{2x^3-3}{x^4-6x} - \dfrac{3}{x-5} - 2$ ← correct
(e) $f'(x) = \dfrac{4x^3-6}{x^4-6x} - \dfrac{3}{x-5} - 2x$

$= \frac{1}{2}\ln(x^4-6x) - 2x - 3\ln(x-5)$

$\Rightarrow f'(x) = \frac{1}{2}\cdot\frac{4x^2-6x}{x^4-6x} - 2 - 3\cdot\frac{1}{x-5}$

$\Rightarrow \boxed{f'(x) = \frac{2x^2-3x}{x^4-6x} - \frac{3}{x-5} - 2}$

19. Determine where the function f **IS** differentiable.

$$f(x) = \begin{cases} 6 - 3x, & x < 0 \\ 2x^2 - 3x + 6, & 0 \le x < 1 \\ 7x - 2x^3, & 1 \le x \le 2 \\ x + x^3, & x > 2 \end{cases}$$

(a) $x = 0$ and $x = 1$ only ← correct

(b) $x = 0$ and $x = 2$ only

(c) $x = 1$ and $x = 2$ only

(d) $x = 1$ only

(e) $x = 0$ only

	continuity	equal?	differentiability	equal?
$\boxed{x=0}$	$6-3(0)=6$ $2(0)^2-3(0)+6=6$	✓	-3 $4(0)-3=-3$	✓
$\boxed{x=1}$	$2(1)^2-3(1)+6=5$ $7(1)-2(1)^3=5$	✓	$4(1)-3=1$ $7-6(1)=1$	✓
~~$x=2$~~	$7(2)-2(2)^3=-2$ $2+(2)^3=10$	✗	not needed	

20. Find the derivative of $f(x) = \arccos(x^3)$.

(a) $\dfrac{-3x^2}{\sqrt{1-x^2}}$

(b) $\dfrac{3x^2}{\sqrt{1-x^2}}$

(c) $\dfrac{-3x^2}{\sqrt{1-x^6}}$ ← correct

(d) $\dfrac{3x^2}{\sqrt{1-x^6}}$

(e) $\dfrac{-1}{\sqrt{1-x^6}}$

$$f'(x) = \frac{-1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 = \boxed{\frac{-3x^2}{\sqrt{1-x^6}}}$$

21. Find the slope of the tangent line to $f(x) = e^{x^2}\ln(x^3+2)$ at $x = -1$.

(a) $e\ln(3) + e$

(b) $3e$ ← correct

(c) $-2e\ln(3) + 3e$

(d) $-2e\ln(3)$

(e) $-6e$

$$f'(x) = 2xe^{x^2}\ln(x^3+2) + \frac{3x^2}{x^3+2}\cdot e^{x^2}$$

$$\Rightarrow f'(-1) = 2(-1)e\,\cancel{\ln(-1+2)}^{\,0} + \frac{3(1)}{-1+2}e = \boxed{3e}$$

22. Find the speed of the particle at time t with position function $\mathbf{r}(t) = \langle 5\sin(t), -3\cos(t)\rangle$.

(a) $\sqrt{25\sin^2(t) + 9\cos^2(t)}$

(b) $\langle -5\cos(t), -3\sin(t)\rangle$

(c) $\sqrt{25\cos^2(t) + 9\sin^2(t)}$ ← correct

(d) $\langle 5\cos(t), 3\sin(t)\rangle$

(e) $\sqrt{34}$

$$\vec{r}\,'(t) = \langle 5\cos(t), 3\sin(t)\rangle$$

$$\Rightarrow \text{speed} = |\vec{r}\,'(t)| = \sqrt{(5\cos(t))^2 + (3\sin(t))^2}$$

$$= \boxed{\sqrt{25\cos^2(t) + 9\sin^2(t)}}$$