

Spring 2005 Math 152

Exam 1B: Solutions

Mon, 21/Feb ©2005, Art Belmonte

For specificity, lengths are in centimeters unless stated otherwise. Graphs appear at the bottom of the right column to conserve space.

1. (d) First compute an antiderivative, then apply the FTC.

• Let $u = \ln x$ $dv = x dx$
 $du = \frac{1}{x} dx$ $v = \frac{1}{2}x^2$. Then

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2.$$

• Hence $\int_1^3 x \ln x dx = \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) \Big|_1^3$
 $= \left(\frac{9}{2} \ln 3 - \frac{9}{4}\right) - \left(-\frac{1}{4}\right) = \frac{9}{2} \ln 3 - 2.$

2. (b) We have

$$dV = A dx = \pi r^2 dx = \pi y^2 dx = \pi (e^x)^2 dx = \pi e^{2x} dx.$$

Therefore

$$V = \int_0^3 \pi e^{2x} dx = \frac{1}{2}\pi e^{2x} \Big|_0^3 = \frac{\pi}{2} (e^6 - 1) \approx 621.13 \text{ cm}^3.$$

3. (e) Find where the curves intersect, draw a sketch, then set up and compute the integral for the volume.

• Now $x^2 = 2x$ implies $x^2 - 2x = 0$ or $x(x - 2) = 0$ whence $x = 0, 2$. (See figure at bottom of column.)

• We have $dV = A dx = (\pi r_o^2 - \pi r_i^2) dx$
 $= \pi ((2x)^2 - (x^2)^2) dx = \pi (4x^2 - x^4) dx.$

• The volume is
 $V = \pi \int_0^2 4x^2 - x^4 dx = \pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right) \Big|_0^2$
 $= \pi \left(\frac{32}{3} - \frac{32}{5}\right) = 32\pi \left(\frac{5-3}{15}\right) = \frac{64}{15}\pi \approx 13.40 \text{ cm}^3.$

4. (c) We have

$$\int_1^2 \frac{x}{x+1} dx = \int_1^2 1 - \frac{1}{x+1} dx = (x - \ln|x+1|) \Big|_1^2$$

$$= (2 - \ln 3) - (1 - \ln 2) = 1 + \ln 2 - \ln 3 = 1 + \ln \frac{2}{3}.$$

5. (e) Via Hooke's Law we have $F(x) = kx$ or $3 = k\left(\frac{1}{6}\right)$, whence $k = 18$. Thus $W = \int_0^2 18x dx = 9x^2 \Big|_0^2 = 36 \text{ ft-lb}$.

6. (a) We have

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{\sqrt{\pi}-0} \int_0^{\sqrt{\pi}} x \sin(x^2) dx$$

$$= \left(-\frac{1}{2\sqrt{\pi}} \cos(x^2)\right) \Big|_0^{\sqrt{\pi}} = \left(\frac{1}{2} \cdot \frac{1}{\sqrt{\pi}}\right) - \left(-\frac{1}{2} \cdot \frac{1}{\sqrt{\pi}}\right) = \frac{1}{\sqrt{\pi}}.$$

7. (d) First compute an antiderivative, then apply the FTC.

• Let $u = x$ $dv = f''(x) dx$
 $du = dx$ $v = f'(x)$. Then

$$\int x f''(x) dx = x f'(x) - \int f'(x) dx = x f'(x) - f(x).$$

• Hence $\int_1^5 x f''(x) dx = (x f'(x) - f(x)) \Big|_1^5$
 $= (5 f'(5) - f(5)) - (f'(1) - f(1))$
 $= (5(2) - 3) - (-1 - 4) = 7 + 5 = 12.$

8. (c) Use half-angle identities.

$$\int_0^{\pi/2} \sin^2 x \cos^2 x dx$$

$$= \int_0^{\pi/2} \frac{1}{2} (1 - \cos 2x) \cdot \frac{1}{2} (1 + \cos 2x) dx$$

$$= \frac{1}{4} \int_0^{\pi/2} 1 - \cos^2 2x dx$$

$$= \frac{1}{4} \int_0^{\pi/2} 1 - \frac{1}{2} (1 + \cos 4x) dx$$

$$= \frac{1}{4} \cdot \frac{1}{2} \int_0^{\pi/2} 1 - \cos 4x dx$$

$$= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x\right) \Big|_0^{\pi/2} = \frac{\pi}{16} - 0 = \frac{\pi}{16}.$$

9. (b) The area is $A = \int_0^2 |x^2 - 1| - 0 dx$, computed thus:

$$A = \int_0^1 1 - x^2 dx + \int_1^2 x^2 - 1 dx$$

$$= \left(x - \frac{1}{3}x^3\right) \Big|_0^1 + \left(\frac{1}{3}x^3 - x\right) \Big|_1^2$$

$$= \left(\frac{2}{3}\right) - (0) + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)$$

$$= \frac{4}{3} + \frac{8}{3} - 2 = 2 \text{ cm}^2.$$

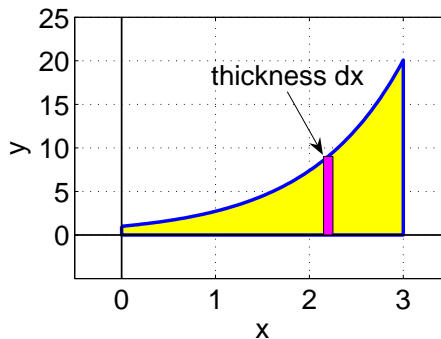
10. (a) Use the trig identity $\sin^2 x + \cos^2 x = 1$.

$$\int \sin^3 x \cos^2 x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx$$

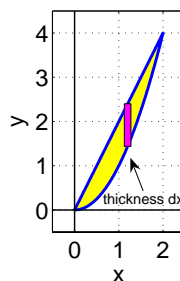
$$= \int (\cos^4 x - \cos^2 x) (-\sin x) dx$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C$$

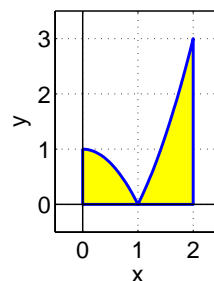
X1B/2



X1B/3



X1B/9



11. A diagram of the tank, its semicircular end, and a rectangular differential layer of water appears at bottom right.

- Clearly the layer is 6 m long, but how wide is it? Look at a diagram of the semicircular end of the tank. The width of the layer is $2y = 2\sqrt{1^2 - z^2} = 2\sqrt{1 - z^2}$.

- The rectangular layer of water has an area of

$$A = LW = 6(2\sqrt{1 - z^2}) = 12\sqrt{1 - z^2}.$$

Its thickness is dz . Here are the volume of the layer, its mass, its weight, and the work required to lift it to the top of the spout. Recall that $\delta = \rho g = 9800 \text{ N/m}^3$.

$$dV = A dz = 12\sqrt{1 - z^2} dz$$

$$dF = \delta dV = 9800(12)\sqrt{1 - z^2} dz$$

$$dW = dF(0 - z) = -9800(12)z\sqrt{1 - z^2} dz$$

- The work required to pump the water out of the tank is

$$\begin{aligned} W &= \int dW = \int_{-1}^0 -9800(12)z(1 - z^2)^{1/2} dz \\ &= (9800)(12)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(1 - z^2)^{3/2} \Big|_{-1}^0 \\ &= (9800)(4) - 0 = 39,200 \text{ J.} \end{aligned}$$

12. The volume of a differential cross-section is

$$dV = A dy = s^2 dy = (2x)^2 dy = 4x^2 dy = 4(1 - \frac{1}{9}y^2) dy.$$

Hence the total volume is

$$\begin{aligned} 4 \int_{-3}^3 1 - \frac{1}{9}y^2 dy &= 8 \int_0^3 1 - \frac{1}{9}y^2 dy \\ &= 8 \left(y - \frac{1}{27}y^3 \right) \Big|_0^3 = 8(3 - 1) - 0 = 16 \text{ cm}^3. \end{aligned}$$

13. Find where the curves intersect, draw a sketch, then set up and compute the integral for the volume.

- Now $\sqrt{x} = 2 - x$ implies $x = 4 - 4x + x^2$ or $x^2 - 5x + 4 = (x - 1)(x - 4) = 0$, whence $x = 1, 4$. (Toss out $x = 4$. See the figure in the next column).
- Via cylindrical shells, we have $V = \int 2\pi rh dy$.

$$\begin{aligned} V &= \int_0^1 2\pi y((2 - y) - (y^2)) dy \\ &= 2\pi \int_0^1 2y - y^2 - y^3 dy \\ &= 2\pi \left(y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4 \right) \Big|_0^1 \\ &= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) - 0 = \frac{5\pi}{6} \approx 2.62 \text{ cm}^3. \end{aligned}$$

14. Let $x = 4 \sin \theta$. Then $dx = 4 \cos \theta d\theta$. Hence (see figure at bottom of the next column)

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{16 - x^2}} &= \int \frac{4 \cos \theta d\theta}{16 \sin^2 \theta \cdot 4 \cos \theta} \\ &= \frac{1}{16} \int \csc^2 \theta d\theta \\ &= -\frac{1}{16} \cot \theta + C \\ &= -\frac{\sqrt{16 - x^2}}{16x} + C \end{aligned}$$

15. We'll integrate the rational function via partial fractions.

- Split the integrand into a sum of partial fractions.

$$\frac{5}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}$$

$$5 = A(x^2 + 4) + (Bx + C)(x - 1)$$

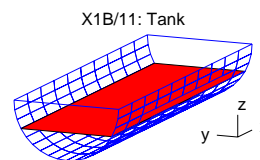
$$0x^2 + 0x + 5 = (A + B)x^2 + (C - B)x + (4A - C)$$

- Thus $A + B = 0$, $C - B = 0$, and $4A - C = 5$. Adding the first two equations we obtain $A + C = 0$, whence $C = -A$. Substituting this into the third equation gives $5A = 5$ whence $A = 1$, $C = -1$, and $B = C = -1$. Therefore,

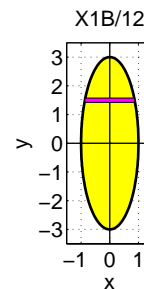
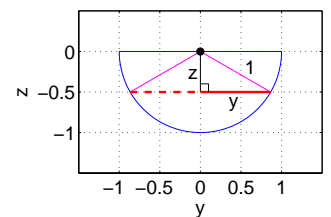
$$\frac{5}{(x - 1)(x^2 + 4)} = \frac{1}{x - 1} - \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4}.$$

- Now integrate term-by-term.

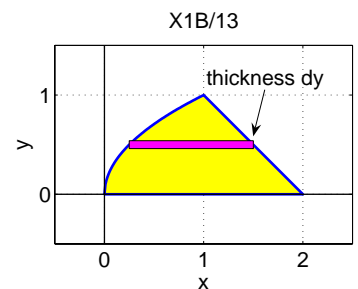
$$\begin{aligned} &\int \frac{5}{(x - 1)(x^2 + 4)} dx \\ &= \int \frac{1}{x - 1} - \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4} dx \\ &= \ln|x - 1| - \frac{1}{2} \ln(x^2 + 4) - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C \end{aligned}$$



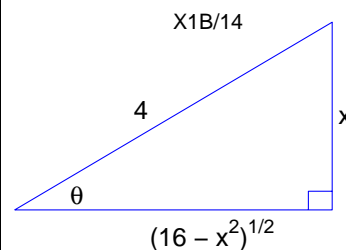
X1B/11: Semicircular back end of tank



X1B/12



X1B/13



X1B/14