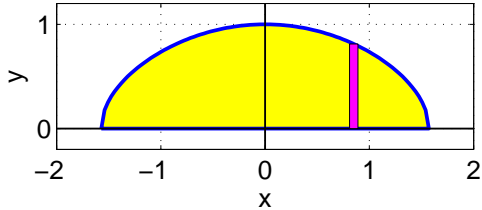


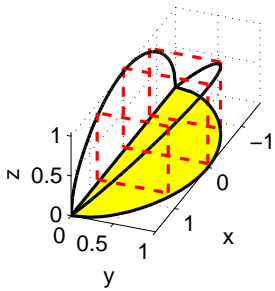
Spring 2007 Math 152
Exam 1B: Problems and Solutions
Mon, 19/Feb ©2007, Art Belmonte

1. (a) The base of a solid is bounded by $y = \sqrt{\cos x}$, $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$, and the x -axis. Each cross-section perpendicular to the x -axis is a square region whose bottom is sitting on this base. Find the volume of the solid.

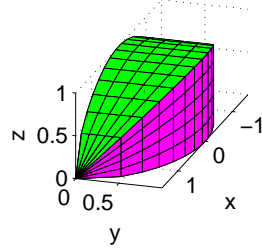
X1B/P1



X1B/P1: 3-D boundaries and cross sections



X1B/P1: 3-D picture of solid; surface patches



- The volume by slicing is

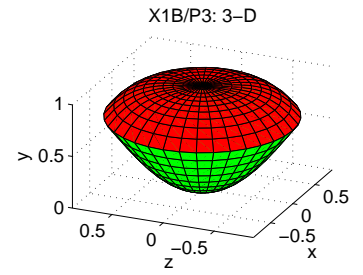
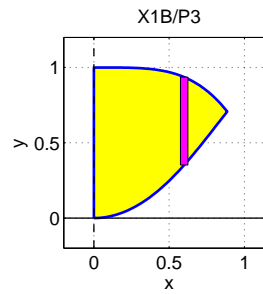
$$\begin{aligned} V &= \int y^2 dx \\ &= \int_{-\pi/2}^{\pi/2} \cos x dx \\ &= \sin x \Big|_{-\pi/2}^{\pi/2} \\ &= 1 - (-1) = 2. \end{aligned}$$

2. (e) Find the average value of $f(x) = \frac{x}{\sqrt{x^2 + 16}}$ on the interval $[0, 3]$.

- The average value is given by

$$\begin{aligned} f_{ave} &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{3-0} \int_0^3 (x^2 + 16)^{-1/2} x dx \\ &= \left(\frac{1}{3}\right) \left(\frac{1}{2}\right) (2) (x^2 + 16)^{1/2} \Big|_0^3 \\ &= \frac{5}{3} - \frac{4}{3} = \frac{1}{3}. \end{aligned}$$

3. (c) The region in the first quadrant bounded by $y = \sin(x^2)$ and $y = \cos(x^2)$, $0 \leq x \leq \frac{1}{2}\sqrt{\pi}$, is revolved about the y -axis. Find the volume of the resulting solid.

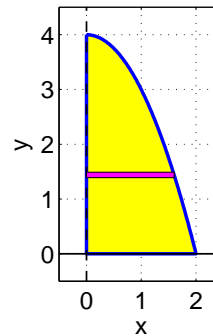


- Via cylindrical shells, the volume swept out by the revolving the stated region about the y -axis is given by

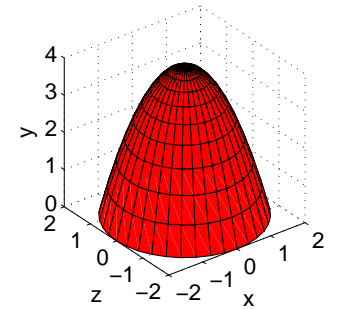
$$\begin{aligned} V &= \int 2\pi rh dx \\ &= \pi \int_0^{\sqrt{\pi}/2} 2x (\cos(x^2) - \sin(x^2)) dx \\ &= \pi (\sin(x^2) + \cos(x^2)) \Big|_0^{\sqrt{\pi}/2} \\ &= \sqrt{2}\pi - \pi \end{aligned}$$

4. (d) Find the volume of the solid generated when the region in the first quadrant bounded by $x = 0$, $y = 0$, and $y = 4 - x^2$ is revolved about the y -axis.

X1B/P4



X1B/P4: 3-D



- Via disks, the volume is

$$\begin{aligned} V &= \int \pi r^2 dy = \int \pi x^2 dy \\ &= \pi \int_0^4 (4 - y) dy \\ &= \left(-\frac{\pi}{2} (4 - y)^2\right) \Big|_0^4 \\ &= 0 - (-8\pi) = 8\pi. \end{aligned}$$

5. (c) Find the value of $\int_0^2 x^3 e^{x^2} dx$.

- Use integration by parts. Compute an antiderivative, then apply the FTC. Let $u = x^2$ and $dv = e^{x^2} x dx$. Then $du = 2x dx$ and $v = \frac{1}{2}e^{x^2}$. Hence $\int x^3 e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \int e^{x^2} x dx = \frac{1}{2}(x^2 - 1)e^{x^2}$.
- Thus $\int_0^2 x^3 e^{x^2} dx = \frac{1}{2}(x^2 - 1)e^{x^2} \Big|_0^2 = \frac{3}{2}e^4 - \left(-\frac{1}{2}\right)$ or $\frac{3e^4 + 1}{2}$.

6. (d) For a certain type of linear spring, the force required to keep it stretched 2 feet beyond its natural length is 5 lb. How much work (in ft-lb) is done stretching this spring 4 feet beyond its natural length?

- Via Hooke's Law, we have $F(x) = kx$ or $5 = 2k$, whence $k = \frac{5}{2}$.

- The work done is $W = \int_a^b F(x) dx$ or

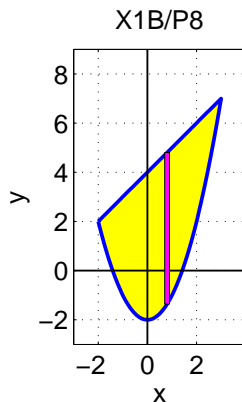
$$\int_0^4 \frac{5}{2}x dx = \frac{5}{4}x^2 \Big|_0^4 = 20 - 0 = 20 \text{ ft-lb.}$$

7. (b) Compute $\int_1^e \frac{\ln x}{x^2} dx$.

- Use integration by parts. Compute an antiderivative, then apply the FTC. Let $u = \ln x$ and $dv = x^{-2} dx$. Then $du = \frac{1}{x} dx$ and $v = -x^{-1}$. Accordingly, $\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int x^{-2} dx = -\frac{1 + \ln x}{x}$.

- Thus $\int_1^e \frac{\ln x}{x^2} dx = \left(-\frac{1 + \ln x}{x}\right) \Big|_1^e = -\frac{2}{e} - (-1)$ or $1 - \frac{2}{e}$.

8. (e) Find the area of the region bounded by the line $y = x + 4$ and the parabola $y = x^2 - 2$.



- When the curves intersect, their y -coordinates are equal. This yields

$$\begin{aligned} x + 4 &= x^2 - 2 \\ 0 &= x^2 - x - 6 \\ 0 &= (x + 2)(x - 3) \\ x &= -2, 3. \end{aligned}$$

- At $x = 0$, an interior point of $[-2, 3]$, the line's y -coordinate is 4, whereas that of the parabola is -2 . Thus the line lies above the parabola.

- Hence the area is given by

$$\begin{aligned} A &= \int_{-2}^3 (x + 4) - (x^2 - 2) dx \\ &= \int_{-2}^3 6 + x - x^2 dx \\ &= \left(6x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) \Big|_{-2}^3 \\ &= \left(18 + \frac{9}{2} - 9\right) - \left(-12 + 2 + \frac{8}{3}\right) \\ &= \frac{27}{2} - \left(-\frac{22}{3}\right) \\ &= \frac{81 + 44}{6} = \frac{125}{6}. \end{aligned}$$

9. (e) A 10-lb object hangs over a ledge at the end of a 20-ft chain that weighs $\frac{1}{2}$ lb per foot. Find the total work (in ft-lb) done hauling the object up to the ledge.

- The work done lifting the object itself is

$$W_{object} = (10)(20) = 200 \text{ ft-lb.}$$

- The work done lifting the rope is

$$W_{rope} = \int_0^{20} \frac{1}{2}x dx = \frac{1}{4}x^2 \Big|_0^{20} = 100 \text{ ft-lb.}$$

- The total amount of work is $200 + 100 = 300$ ft-lb.

10. (b) Compute $\int_0^2 \frac{5x + 7}{x^2 + 4x + 3} dx$.

- Split the rational integrand into a sum of partial fractions.

$$\begin{aligned} \frac{5x + 7}{(x + 1)(x + 3)} &= \frac{A}{x + 1} + \frac{B}{x + 3} \\ 5x + 7 &= A(x + 3) + B(x + 1) \\ 5x + 7 &= (A + B)x + (3A + B) \end{aligned}$$

- Equate coefficients of like terms.

$$\begin{aligned} A + B &= 5 \\ 3A + B &= 7 \end{aligned}$$

Subtracting the second equation from the first gives $-2A = -2$ or $A = 1$. Thus $B = 5 - A = 4$.

- Integrate term-by-term.

$$\begin{aligned} &\int_0^2 \frac{1}{x + 1} + \frac{4}{x + 3} dx \\ &= (\ln|x + 1| + 4 \ln|x + 3|) \Big|_0^2 \\ &= (\ln 3 + 4 \ln 5) - (\ln 1 + 4 \ln 3) \\ &= 4 \ln 5 - 3 \ln 3 \end{aligned}$$

Here we have recalled that $\ln 1 = 0$.

11. Compute $\int \cos t \cos 4t \, dt$.

- Use a trigonometric product formula

$$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$$

plus the fact the the cosine function is even:
 $\cos(-\theta) = \cos \theta$.

- Integrate the transformed integral as follows.

$$\begin{aligned} \int \cos t \cos 4t \, dt &= \int \frac{1}{2}(\cos(-3t) + \cos 5t) \, dt \\ &= \frac{1}{2} \int \cos 3t + \cos 5t \, dt \\ &= \frac{1}{2} \left(\frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t \right) + C \\ &\text{or } \frac{1}{6} \sin 3t + \frac{1}{10} \sin 5t + C \end{aligned}$$

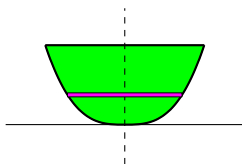
12. Find $\int \cos^3 3\theta \sin^{-2} 3\theta \, d\theta$.

- Use the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned} &\int \cos 3\theta (1 - \sin^2 3\theta) \sin^{-2} 3\theta \, d\theta \\ &= \int (\sin 3\theta)^{-2} \cos 3\theta - \cos 3\theta \, d\theta \\ &= -\frac{1}{3} (\sin 3\theta)^{-1} - \frac{1}{3} \sin 3\theta + C \\ &\text{or } -\frac{1}{3} (\csc 3\theta + \sin 3\theta) + C \end{aligned}$$

13. A tank full of water has the depicted shape of a solid of revolution obtained by rotating about the y -axis the region in the first quadrant bounded by $x = (4y)^{1/3}$ and the lines $y = 2$ and $x = 0$. Find the work (in joules) required to pump the water out of the tank. Lengths are in meters. The density of water is $\rho = 1000 \text{ kg/m}^3$ and $g = 9.8 \text{ m/s}^2$.

Front view of tank filled with water with a typical water layer



- From the diagram on the exam depicting a typical layer of water, we follow the “march of the differentials” for volume, mass, force, and work.

$$\begin{aligned} dV &= \pi r^2 dy = \pi x^2 dy = \pi (4y)^{2/3} dy \\ dm &= \rho dV = \rho \pi (4y)^{2/3} dy \\ dF &= (dm)g = g\rho \pi (4y)^{2/3} dy \\ dW &= (dF)(D) = g\rho \pi (4y)^{2/3} (2 - y) dy \end{aligned}$$

- Set up the and dispatch the work integral. (This begs for calculator firepower. Perhaps in future terms. . .)

$$\begin{aligned} W &= \int dW = g\rho \pi \int_0^2 2(4)^{2/3} y^{2/3} - 4^{2/3} y^{5/3} dy \\ &= g\rho \pi \left(2(4)^{2/3} \left(\frac{3}{5} \right) y^{5/3} - 4^{2/3} \left(\frac{3}{8} \right) y^{8/3} \right) \Big|_0^2 \\ &= g\rho \pi \left(\frac{48}{5} - \frac{48}{8} \right) - 0 = 48g\rho \pi \left(\frac{8-5}{40} \right) \\ &= \frac{18}{5} g\rho \pi = \frac{18}{5} \left(\frac{98}{10} \right) (1000) \pi = 35,280\pi \text{ joules} \end{aligned}$$

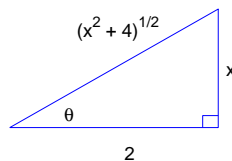
14. Evaluate $\int \frac{x + 1}{(x^2 + 4)^{3/2}} dx$.

- Use trigonometric substitution. Let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta \, d\theta$.
- Changing variables, the integral becomes

$$\begin{aligned} &\int \frac{2 \tan \theta + 1}{8 \sec^3 \theta} \cdot 2 \sec^2 \theta \, d\theta \\ &= \int \frac{1}{2} \sin \theta + \frac{1}{4} \cos \theta \, d\theta \\ &= -\frac{1}{2} \cos \theta + \frac{1}{4} \sin \theta + C. \text{ [continued...]} \end{aligned}$$

- Finally, we express the antiderivative in terms of the original variable x .

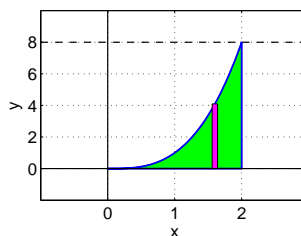
X1B/P14



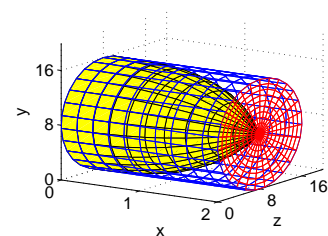
$$\begin{aligned} &-\frac{1}{2} \frac{2}{\sqrt{x^2 + 4}} + \frac{1}{4} \frac{x}{\sqrt{x^2 + 4}} + C \\ &= \frac{x}{4\sqrt{x^2 + 4}} - \frac{1}{\sqrt{x^2 + 4}} + C \\ &\text{or } \frac{x - 4}{4\sqrt{x^2 + 4}} + C \end{aligned}$$

15. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = x^3$, the line $x = 2$, and the x -axis, about the line $y = 8$.

X1B/P15



X1B/P15: 3-D



- Via washers, the volume is

$$\begin{aligned} V &= \int \pi r_o^2 - \pi r_i^2 dx \\ &= \pi \int_0^2 8^2 - (8 - x^3)^2 dx \\ &= \pi \int_0^2 16x^3 - x^6 dx \\ &= \pi \left(4x^4 - \frac{1}{7}x^7 \right) \Big|_0^2 \\ &= \left(64 - \frac{128}{7} \right) \pi - 0 = \frac{320}{7} \pi \end{aligned}$$