Spring 2007 Math 152
Exam 2A: Problems and Solutions
Mon, 26/Mar ©2007, Art Belmonte

1. (a) Find $\int_0^{\infty} xe^{-x}\,dx$.

   - First compute the indefinite integral $\int xe^{-x}\,dx$ via integration by parts. Let $u = x$ and $dv = e^{-x}\,dx$. Then $du = dx$ and $v = -e^{-x}$. Accordingly, we have $\int xe^{-x}dx = -xe^{-x} + \int e^{-x}\,dx = -(x+1)\,e^{-x}$.
   - Therefore,

$$\begin{aligned}\int_0^{\infty} xe^{-x}\,dx &= \lim_{t\to\infty}\int_0^t xe^{-x}\,dx\\ &= \lim_{t\to\infty}\left(-(x+1)\,e^{-x}\right)\Big|_0^t\\ &= \lim_{t\to\infty}\left(-\frac{t+1}{e^t}+1\right) = 1,\end{aligned}$$

   since $\lim_{t\to\infty}\dfrac{t+1}{e^t} \overset{\text{L'H}}{=} \lim_{t\to\infty}\dfrac{1}{e^t} = 0$.

2. (b) A tank contains 200 L of brine, initially with 35 kg of salt in solution. Brine containing 0.2 kg/L of salt enters the tank at the rate of 5 L/min. It flows out of the tank at the same rate. The mixture in the tank is kept uniform by constant stirring. If $y(t)$ is the amount of salt in the tank at time t, which of the following initial value problems models the behavior of $y(t)$?

   - The classical balance law states that net rate = rate-in − rate-out.

$$\begin{aligned}\frac{dy}{dt} &= \left(0.2\frac{\text{kg}}{\text{L}}\right)\left(5\frac{\text{L}}{\text{min}}\right) - \left(\frac{y\ \text{kg}}{200\ \text{L}}\right)\left(5\frac{\text{L}}{\text{min}}\right)\\ \frac{dy}{dt} &= 1 - \frac{y}{40}\end{aligned}$$

   - The initial condition is $y(0) = 35$.

3. (b) Point masses of mass 7, 10, and 13 are positioned at $(-3, 2)$, $(3, 5)$, and $(4, 3)$, respectively. Find the center of mass of the system.

   - Let $\mathbf{p} = \begin{bmatrix} 7 & 10 & 13 \end{bmatrix}$ be a row vector of the masses and $\mathbf{r} = \begin{bmatrix} -3 & 2 \\ 3 & 5 \\ 4 & 3 \end{bmatrix}$ a matrix whose rows are position vectors of the coordinates. Let m be the sum of the masses: $7 + 10 + 13 = 30$.
   - Then the position vector of the center of mass is given by $\text{CM} = [\bar{x}, \bar{y}] = \dfrac{1}{m}\mathbf{pr}$, as follows.

$$\begin{aligned}\frac{1}{m}\mathbf{pr} &= \frac{1}{30}\begin{bmatrix} 7 & 10 & 13 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 3 & 5 \\ 4 & 3 \end{bmatrix}\\ &= \frac{1}{30}[-21+30+52,\ 14+50+39]\\ &= [61/30,\ 103/30]\end{aligned}$$

4. (e) Solve the differential equation $y' + y\tan x = \sec x$.

   - The differential equation $y' + y\tan x = \sec x$ is in standard linear form (SLF).
   - An integrating factor is $\mu = \exp\left(\int \tan x\,dx\right) = \exp\left(\int \frac{\sin x}{\cos x}\,dx\right) = \exp\left(-\ln(\cos x)\right) = \sec x$.
   - Multiplying the SLF by the integrating factor $\mu = \sec x$, we obtain $y'\sec x + y\sec x\tan x = \sec^2 x$ or $(y\sec x)' = \sec^2 x$.
   - Thus $y\sec x = \tan x + K$, whence $y = \sin x + K\cos x$.

5. (e) Solve the differential equation $y'/x = e^{x-y}$.

   - The differential equation $y'/x = e^{x-y} = e^x/e^y$ is separable.
   - Thus $e^y\,dy = xe^x\,dx$ implies $e^y = (x-1)\,e^x + C$, whence $y = \ln\left((x-1)\,e^x + C\right)$.
   - Note that we used integration by parts to compute $\int xe^x\,dx$. Indeed, let $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. Therefore,

$$\int xe^x\,dx = xe^x - \int e^x\,dx = (x-1)\,e^x.$$

6. (e) Find the area of the surface generated by rotating the Cartesian curve $y = x^3/3$, $0 \le x \le \sqrt[4]{15}$, about the x-axis.

   - The surface area is given by $S = \int 2\pi r\,ds$.
   - We have

$$\begin{aligned}\int 2\pi r\,ds &= \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ &= \int_0^{\sqrt[4]{15}} 2\pi\left(x^3/3\right)\sqrt{1+\left(x^2\right)^2}\,dx\\ &= \frac{2\pi}{3}\int_0^{\sqrt[4]{15}}\left(1+x^4\right)^{1/2}x^3\,dx\\ &= \frac{2\pi}{3}\,\frac{1}{4}\int_0^{\sqrt[4]{15}}\left(1+x^4\right)^{1/2}4x^3\,dx\\ &= \frac{\pi}{6}\left(\frac{2}{3}\right)\left(1+x^4\right)^{3/2}\Big|_0^{\sqrt[4]{15}}\\ &= \frac{\pi}{9}(64) - \frac{\pi}{9}(1) = \frac{63}{9}\pi = 7\pi.\end{aligned}$$

7. (c) Find the arc length of the curve $x = t^3/3$, $y = t^2/2$, $\sqrt{3} \le t \le \sqrt{24}$.

   - The arc length is $L = \int ds$. [continued on reverse]

- We have

$$\begin{aligned}\int ds &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_{\sqrt{3}}^{\sqrt{24}} \sqrt{\left(t^2\right)^2 + (t)^2}\, dt \\ &= \int_{\sqrt{3}}^{\sqrt{24}} \sqrt{t^4 + t^2}\, dt \\ &= \int_{\sqrt{3}}^{\sqrt{24}} \left(t^2+1\right)^{1/2} t\, dt \\ &= \frac{1}{2}\left(\frac{2}{3}\right)\left(t^2+1\right)^{3/2}\Big|_{\sqrt{3}}^{\sqrt{24}} \\ &= \frac{1}{3}(125) - \frac{1}{3}(8) = \frac{117}{3} = 39.\end{aligned}$$

8. (d) Does $\displaystyle\int_1^\infty \frac{e^{-x}}{x}\, dx$ converge?

- Observe that $0 < \dfrac{e^{-x}}{x} \leq e^{-x}$ on $[1, \infty)$.
- Moreover,

$$\begin{aligned}\int_1^\infty e^{-x}\, dx &= \lim_{t\to\infty} \int_1^t e^{-x}\, dx \\ &= \lim_{t\to\infty} \left(-e^{-x}\right)\Big|_1^t \\ &= \lim_{t\to\infty} \left(-e^{-t} + \frac{1}{e}\right) = \frac{1}{e} < 1.\end{aligned}$$

- By a comparison theorem, we conclude that $\displaystyle\int_1^\infty \frac{e^{-x}}{x}\, dx$ converges to a value L between 0 and 1.

9. Taking into account anything that might make the integral improper, find $\displaystyle\int_{-1}^8 x^{-4/3}\, dx$.

- Now $f(x) = x^{-4/3}$ has an infinite discontinuity at $x = 0$. Accordingly,

$$\int_{-1}^8 x^{-4/3}\, dx = \int_{-1}^0 x^{-4/3}\, dx + \int_0^8 x^{-4/3}\, dx.$$

Since both $\int_{-1}^0 x^{-4/3}\, dx$ and $\int_0^8 x^{-4/3}\, dx$ diverge to ∞, we conclude that $\int_{-1}^8 x^{-4/3}\, dx$ diverges to ∞; details follow.

- We have

$$\begin{aligned}\int_{-1}^0 x^{-4/3}\, dx &= \lim_{t\to 0^-} \int_{-1}^t x^{-4/3}\, dx \\ &= \lim_{t\to 0^-} \left(-3x^{-1/3}\right)\Big|_{-1}^t \\ &= \lim_{t\to 0^-} \left(\frac{-3}{\sqrt[3]{t}} - 3\right) = \infty.\end{aligned}$$

- Similarly,

$$\begin{aligned}\int_0^8 x^{-4/3}\, dx &= \lim_{t\to 0^+} \int_t^8 x^{-4/3}\, dx \\ &= \lim_{t\to 0^+} \left(-3x^{-1/3}\right)\Big|_t^8 \\ &= \lim_{t\to 0^+} \left(-\frac{3}{2} + \frac{3}{\sqrt[3]{t}}\right) = \infty.\end{aligned}$$

10. Consider the curve $y = f(x) = \cos(x/8)$ on the interval $0 \leq x \leq 4$.

(a) Use the Trapezoidal Rule with 4 intervals to estimate $\displaystyle\int_0^4 \cos(x/8)\, dx$. You may use the following table.

x	0	1	2	3	4
$\cos(x/8)$	1.0000	0.9922	0.9689	0.9305	0.8776

- The Trapezoidal Rule gives

$$\begin{aligned}T_4 &= 1\Big(\frac{1.0000 + 0.8776}{2} \\ &\quad + 0.9922 + 0.9689 + 0.9305\Big) \\ &= 3.8304.\end{aligned}$$

(b) The difference between the actual value of an integral and the number obtained by using the Trapezoidal Rule with n intervals is at most

$$\frac{K(b-a)^3}{12n^2} \qquad (*)$$

where K is the maximum of $\left|f''(x)\right|$ on the interval of integration. Determine a reasonable choice for n so that if n intervals were used instead of 4 as in part (a), the computed value of $\int_0^4 \cos(x/8)\, dx$ would be guaranteed by (*) to be within 10^{-8} of the actual value.

- Now $f''(x) = -\frac{1}{64}\cos(x/8)$, whence $K = \left|f''(0)\right| = \frac{1}{64}$. Thus $\dfrac{\frac{1}{64}(4-0)^3}{12n^2} < \dfrac{1}{10^8}$ implies $n > \sqrt{\dfrac{10^8}{12}} \approx 2886.75$. So choose $n = 2887$. [The final inequality sufficed.]

11. Find the centroid (center of mass of a region of constant density) of the region in the first quadrant bounded by $y = \frac{3}{2}x^2$, the x-axis, and the line $x = 4$. [See ***Notes*** at end!]

- Let $\delta = k$ be the constant density of the region D. The mass of the region is $m = \iint_D \delta\, dA$, which we now compute.

$$\begin{aligned}m = \iint_D \delta\, dA &= \int_0^4 \int_0^{3x^2/2} k\, dy\, dx \\ &= \int_0^4 ky\Big|_{y=0}^{y=3x^2/2} dx \\ &= \int_0^4 \left(\frac{3kx^2}{2} - 0\right) dx \\ &= \frac{kx^3}{2}\Big|_0^4 = 32k - 0 = 32k\end{aligned}$$

- The position vector of the center of mass of the region is $\mathrm{CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \delta [x, y] \, dA$. We have

$$\begin{aligned}
& \frac{1}{m} \iint_D \delta [x, y] \, dA \\
= & \frac{1}{32k} \int_0^4 \int_0^{3x^2/2} k [x, y] \, dy \, dx \\
= & \frac{1}{32} \left[\int_0^4 \int_0^{3x^2/2} x \, dy \, dx, \int_0^4 \int_0^{3x^2/2} y \, dy \, dx \right] \\
= & \frac{1}{32} \left[\int_0^4 xy \Big|_{y=0}^{y=3x^2/2} dx, \int_0^4 \tfrac{1}{2} y^2 \Big|_{y=0}^{y=3x^2/2} dx \right] \\
= & \frac{1}{32} \left[\frac{3}{2} \int_0^4 x^3 \, dx, \frac{9}{8} \int_0^4 x^4 \, dx \right] \\
= & \frac{1}{32} \left[\tfrac{3}{8} x^4 \Big|_0^4, \tfrac{9}{40} x^5 \Big|_0^4 \right] \\
= & \frac{1}{32} \left[\frac{3 (2)^8}{8}, \frac{9 (2)^{10}}{40} \right] \\
= & \left[3, \frac{36}{5} \right] = [3, 7.2] .
\end{aligned}$$

12. An aquarium has an odd-shaped window that opens to public view. The water level in the aquarium is level with the top of the window, which ranges from floor level at its middle to 4 meters high at the top. The window is bounded by the curve $y = \frac{1}{4} x^4$, $-2 \leq x \leq 2$, and the line $y = 4$, as shown below. Find the hydrostatic force pushing against the window. Water has a density of $\rho = 1000 \text{ kg/m}^3$; also, $g = 9.8 \text{ m/s}^2$.

   - The window is a vertical plate at variable depth. Build up the integral for hydrostatic force step by step.

$$\begin{aligned}
P &= \rho g \, (\text{depth}) = \rho g \, (4 - y) \\
dA &= w \, dy = 2x \, dy = 2 \, (4y)^{1/4} \, dy \\
dF &= P \, dA = 2\rho g \, (4y)^{1/4} \, (4 - y) \, dy \\
F &= \int_0^4 2\rho g \, (4y)^{1/4} \, (4 - y) \, dy \\
&= 2\rho g \, (4)^{1/4} \int_0^4 4y^{1/4} - y^{5/4} \, dy \\
&= 2\rho g \, (4)^{1/4} \left(\frac{16}{5} y^{5/4} - \frac{4}{9} y^{9/4} \right) \Big|_0^4 \\
&= 2\rho g \left(4^{6/4} \left(\frac{16}{5} \right) - 4^{10/4} \left(\frac{4}{9} \right) \right) - 0 \\
&= 2 \left(\frac{2^7}{5} - \frac{2^7}{9} \right) \rho g = 2^8 \left(\frac{9 - 5}{45} \right) \rho g = \frac{2^{10}}{45} \rho g \\
&= \frac{1024}{45} \, (1000) \left(\frac{98}{10} \right) \\
&= \frac{2{,}007{,}040}{9} \approx 2.23 \times 10^5 \text{ N}
\end{aligned}$$

Notes

Here is another way to compute the center of mass in Problem 11.

- The mass is

$$\begin{aligned}
m &= \int_a^b \delta \, (f \, (x) - g \, (x)) \, dx \\
&= \int_0^4 k \left(\tfrac{3}{2} x^2 - 0 \right) dx \\
&= \tfrac{1}{2} k x^3 \Big|_0^4 = 32k - 0 = 32k.
\end{aligned}$$

- The moment with respect to the x-axis is

$$\begin{aligned}
M_y &= \int_a^b \delta x \, (f \, (x) - g \, (x)) \, dx \\
&= \int_0^4 kx \left(\tfrac{3}{2} x^2 - 0 \right) dx \\
&= \int_0^4 \tfrac{3}{2} k x^3 \, dx \\
&= \tfrac{3}{8} k x^4 \Big|_0^4 = 96k - 0 = 96k.
\end{aligned}$$

- The moment with respect to the y-axis is

$$\begin{aligned}
M_x &= \int_a^b \delta \left(\tfrac{1}{2} f^2 \, (x) - \tfrac{1}{2} g^2 \, (x) \right) dx \\
&= \int_0^4 \tfrac{1}{2} k \left(\left(\tfrac{3}{2} x^2 \right)^2 - 0 \right) dx \\
&= \int_0^4 \tfrac{9}{8} k x^4 \, dx \\
&= \tfrac{9}{40} k x^5 \Big|_0^4 = \tfrac{9 (2)^7}{5} k.
\end{aligned}$$

- Accordingly, the center of mass is

$$\begin{aligned}
\mathrm{CM} = [\bar{x}, \bar{y}] &= \left[\frac{M_y}{m}, \frac{M_x}{m} \right] \\
&= \left[\frac{96k}{32k}, \frac{9 \, (2)^7 \, k/5}{2^5 k} \right] \\
&= \left[3, \frac{36}{5} \right] = [3, 7.2] .
\end{aligned}$$