

1. (b) Given $y'/x = e^{x-y} = e^x/e^y$, separate variables then antidifferentiate and isolate y to obtain $y = \ln((x-1)e^x + C)$.

2. (e) Note that $0 < \int_1^\infty \frac{e^{-x}}{x} dx \leq \int_1^\infty e^{-x} dx = \frac{1}{e} \approx 0.37 < 1$. By a comparison theorem, we conclude that

$$\int_1^\infty \frac{e^{-x}}{x} dx \text{ converges to a value } L \text{ between } 0 \text{ and } 1.$$

3. (c) The arc length is $\int_{\sqrt{3}}^{\sqrt{24}} \sqrt{(t^2)^2 + (t)^2} dt = 39$.

4. (c) $\frac{dy}{dt} = 1 - \frac{y}{40}$, $y(0) = 35$ [classical balance law: net rate = rate-in - rate-out]

5. (b) $\int_0^\infty x e^{-x} dx = 1$ [improper integral; integration by parts]

6. (c) The differential equation $y' + y \tan x = \sec x$ is in standard linear form (SLF). Multiply by the integrating factor $\mu = \sec x$, then antidifferentiate and isolate y to obtain $y = \sin x + K \cos x$.

7. (e) Let $\mathbf{p} = [7 \ 10 \ 13]$ be a row vector of the masses and $\mathbf{r} = \begin{bmatrix} -3 & 2 \\ 3 & 5 \\ 4 & 3 \end{bmatrix}$ a matrix whose rows are position vectors of the coordinates. Let $m = \text{sum}(\mathbf{p}) = 30$. Then the position vector of the center of mass is given by $\text{CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \mathbf{p} \mathbf{r} = [61/30, 103/30]$.

8. (a) The surface area is $\int_0^{\sqrt[4]{15}} 2\pi (x^3/3) \sqrt{1 + (x^2)^2} dx = 7\pi$.

9. The window is a vertical plate at variable depth. Build up the integral for hydrostatic force step by step.

$$\begin{aligned} P &= \rho g (\text{depth}) = \rho g (4 - y) \\ dA &= w dy = 2x dy = 2(4y)^{1/4} dy \\ dF &= P dA = 2\rho g (4y)^{1/4} (4 - y) dy \\ F &= \int_0^4 2\rho g (4y)^{1/4} (4 - y) dy \\ &= \frac{1024}{45} \rho g = \frac{2,007,040}{9} \approx 2.23 \times 10^5 \text{ N} \end{aligned}$$

10. Let $\delta = k$ be the constant density of the region D . The mass of the region is

$$m = \iint_D \delta dA = \int_0^4 \int_0^{3x^2/2} k dy dx = 32k. \text{ The position vector of the center of mass of the region is}$$

$$\text{CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \delta [x, y] dA = \frac{1}{32k} \int_0^4 \int_0^{3x^2/2} k [x, y] dy dx = \left[3, \frac{36}{5}\right] = [3, 7.2].$$

Naturally, students may compute the center of mass using moments as in Section 9.6.

11. (a) The Trapezoidal Rule gives $T_4 = 1 \left(\frac{1}{2} (1.0000 + 0.8776) + 0.9922 + 0.9689 + 0.9305 \right) = 3.8304$.

(b) Now $f''(x) = -\frac{1}{64} \cos(x/8)$, whence $K = |f''(0)| = \frac{1}{64}$. Thus $\frac{1}{64} \frac{(4-0)^3}{12n^2} < \frac{1}{10^8}$ implies

$$n > \sqrt{10^8/12} \approx 2886.75; \text{ for example, choose } n = 2887 \text{ or } 3000. \text{ (The final inequality suffices.)}$$

12. Now $f(x) = x^{-4/3}$ has an infinite discontinuity at $x = 0$. Accordingly, $\int_{-1}^8 x^{-4/3} dx = \int_{-1}^0 x^{-4/3} dx + \int_0^8 x^{-4/3} dx$. Since both $\int_{-1}^0 x^{-4/3} dx$ and $\int_0^8 x^{-4/3} dx$ diverge to ∞ , we conclude that $\int_{-1}^8 x^{-4/3} dx$ diverges to ∞ .

Spring 2007 Math 152
Exam 2: Executive Summary
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Chapters and Sections

8: Techniques of Integration

- 8.8 Approximate Integration
- 8.9 Improper Integrals

9: Further Applications of Integration

- 9.1 [Separable First-Order] Differential Equations
- 9.2 First-Order Linear Equations
- 9.3 Arc Length
- 9.4 Area of a Surface of Revolution
- 9.5 Moments and Centers of Mass
- 9.6 Hydrostatic Pressure and Force

Fundamental Concepts

Legend

$$h = \Delta x = \frac{b-a}{n} \quad dA = dx dy = dy dx \quad K = \max_{a \leq x \leq b} |f''(x)|$$

$$x_k = a + kh; k = 0, \dots, n \quad z: \text{depth} \quad M = \max_{a \leq x \leq b} |f^{(4)}(x)|$$

$$8.8 \quad L_n = h \sum_{k=0}^{n-1} f(x_k) \quad R_n = h \sum_{k=1}^n f(x_k)$$

$$M_n = h \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right)$$

$$T_n = h \left(\frac{f(x_0) + f(x_n)}{2} + \sum_{k=1}^{n-1} f(x_k) \right)$$

$$S_n = \frac{h}{3} \left(f(x_0) + f(x_n) + 4 \sum_{i=0}^{n/2-1} f(x_{2i+1}) + 2 \sum_{j=1}^{n/2-1} f(x_{2j}) \right)$$

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2} \quad |E_{M_n}| \leq \frac{K(b-a)^3}{24n^2}$$

$$|E_{S_n}| \leq \frac{M(b-a)^5}{180n^4} \text{ [For Simpson's Rule, } n \text{ is even!]}$$

8.9 Use *limits* and/or comparison theorems to compute improper integrals.

9.1 $dy/dx = f(x, y) \implies g(y) dy = h(x) dx \implies$
 $G(y) = H(x) + C \implies y = G^{-1}(H(x) + C)$

9.2 $y' + p(x)y = q(x)$
 $\mu(x) = \exp\left(\int p(x) dx\right)$
 $\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x)$
 $(\mu(x)y)' = \mu(x)q(x)$
 $y = \frac{1}{\mu(x)} \left(\int \mu(x)q(x) dx + C \right)$

9.3 $L = \int ds \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
 $= \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

9.4 $S = \int 2\pi r ds$

9.5 Let $\mathbf{p} = [m_1 \ m_2 \ \dots \ m_n]$, $m = \sum_{k=1}^n m_k$, and

$$\mathbf{r} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}. \text{ Then CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \mathbf{p}\mathbf{r}.$$

Let a planar region D have density δ . Then its mass is $m = \iint_D \delta dA$ and its center of mass is given by $\text{CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \delta [x, y] dA$.

9.6 $P = \delta z = \rho g z$, $dA = w dy$, $dF = P dA$, $F = \int dF$

Additional Details

- $\int_a^\infty \frac{1}{x^p} dx$ converges for $p > 1$ and diverges for $p \leq 1$.
- Comparison theorems for improper integrals: Let f and g be continuous on (a, ∞) with $f \geq g \geq 0$. [Just think of areas under curves above the x -axis and the following assertions are clear.]
 1. If $\int_a^\infty f(x) dx$ converges, then so does $\int_a^\infty g(x) dx$. Moreover, $0 \leq \int_a^\infty g(x) dx \leq \int_a^\infty f(x) dx = L$.
 2. If $\int_a^\infty g(x) dx$ diverges, then so does $\int_a^\infty f(x) dx$. Moreover, $\int_a^\infty g(x) dx = \int_a^\infty f(x) dx = \infty$.
- If a differential equation has an initial condition, $y(x_0) = y_0$, then this initial condition is used to resolve the constant involved in the general solution to the differential equation using algebra.

General framework for an [improper] integral

Let $(a, b) \subset \mathbb{R}$. Here a is either a real number or $-\infty$ and b is either a real number or ∞ . Label $x_0 = a$ and $x_m = b$. Let $\{x_i\}_{i=1}^{m-1}$ be real numbers such that $x_0 < x_1 < \dots < x_m$ and define $D = (a, b) \setminus \{x_i\}_{i=1}^{m-1}$. [In other words, D is (a, b) with the points x_1, \dots, x_{m-1} removed.] Suppose that $f: D \rightarrow \mathbb{R}$ is continuous and that f is discontinuous at x_1, \dots, x_{m-1} . Finally, let $c_i \in (x_{i-1}, x_i)$, $i = 1, \dots, m$. [Note that f is continuous at the interior points c_i .] Then we define

$$\int_a^b f(x) dx = \sum_{i=1}^m \left(\lim_{t \rightarrow x_{i-1}^+} \int_t^{c_i} f(x) dx + \lim_{t \rightarrow x_i^-} \int_{c_i}^t f(x) dx \right),$$

provided that *all* these limits exist.