

Spring 2007 Math 152

Exam 3A: Problems and Solutions

Fri, 04/May

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1. (c) Find the vector projection of $\mathbf{b} = -\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ onto $\mathbf{a} = -4\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$.

- The vector projection is

$$\begin{aligned} \text{proj}_{\mathbf{a}} \mathbf{b} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \right) \mathbf{a} \\ &= \left(\frac{4 - 16 - 28}{\sqrt{16 + 64 + 49}} \right) \frac{[-4, 8, 7]}{\sqrt{129}} \\ &= -\frac{40}{129} [-4, 8, 7] \\ &= \frac{160}{129} \mathbf{i} - \frac{320}{129} \mathbf{j} - \frac{280}{129} \mathbf{k}. \end{aligned}$$

2. (b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{5^n}$.

- The sum of geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right) \left(-\frac{3}{5}\right)^{n-1}$ is

$$\frac{1/5}{1 - (-3/5)} = \frac{1/5}{8/5} = \frac{1}{8}.$$

3. (e) Find the cosine of the angle θ between the vectors $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

- The cosine is

$$\begin{aligned} \cos \theta &= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \\ &= \frac{2 + 12 - 12}{\sqrt{4 + 36 + 16} \sqrt{1 + 4 + 9}} \\ &= \frac{2}{\sqrt{4 \cdot 14^2}} = \frac{2}{2(14)} = \frac{1}{14}. \end{aligned}$$

4. (d) Which of the following statements is true for the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$?

- The series is convergent, but not absolutely convergent, as we now show. (We say that the series is *conditionally convergent*.)

- This alternating series converges by the Alternating Series Test since $b_n = |a_n| = \frac{1}{n \ln n} \downarrow 0$.

- However, the corresponding series of absolute values

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges by the Integral Test since } \int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} (\ln(\ln t) - \ln(\ln 2)) = \infty.$$

5. (a) The series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test, as we now show.

- First compute an antiderivative using integration by parts. Let $u = \ln x$ and $dv = x^{-2} dx$. Then $du = x^{-1} dx$ and $v = -x^{-1}$. Hence

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int x^{-2} dx = -\frac{(1 + \ln x)}{x}.$$

- Accordingly,

$$\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \left(-\frac{1 + \ln t}{t} + \frac{1 + \ln 2}{2} \right) = \frac{1 + \ln 2}{2}$$

since $\lim_{t \rightarrow \infty} \frac{1 + \ln t}{t} = \lim_{t \rightarrow \infty} \frac{1/t}{1} = 0$ by employing L'Hospital's Rule.

6. (c) Find $T_2(x)$, the second-degree Taylor polynomial for $f(x) = \sqrt{x+3}$ about $a = 1$.

- Compute the first two derivatives of f .

$$\begin{aligned} f(x) &= (x+3)^{1/2} \\ f'(x) &= \frac{1}{2}(x+3)^{-1/2} \\ f''(x) &= -\frac{1}{4}(x+3)^{-3/2} \end{aligned}$$

- Evaluate f and these derivatives at 1.

$$\begin{aligned} f(1) &= 2 \\ f'(1) &= \frac{1}{4} \\ f''(1) &= -\frac{1}{32} \end{aligned}$$

- Write down the requested Taylor polynomial.

$$T_2(x) = \sum_{n=0}^2 \frac{f^{(n)}(1)}{n!} (x-1)^n = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$$

7. (b) For what values of x does the power series

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} (x-3)^n \text{ converge?}$$

- The Root Test guarantees absolute convergence

$$\text{provided } \sqrt[n]{|a_n|} = \frac{\sqrt[n]{n^2} |x-3|}{2} \rightarrow \frac{|x-3|}{2} < 1,$$

whence $|x-3| < 2$. Thus $-2 < x-3 < 2$ or

$1 < x < 5$. Alternatively, the Ratio Test also yields

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2 |x-3|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2 |x-3|^n} = \frac{(1 + \frac{1}{n})^2 |x-3|}{2} \rightarrow \frac{|x-3|}{2} < 1.$$

- For $x = 1$, the series is $\sum (-1)^n n^2$, which diverges by the Test for Divergence.

- For $x = 5$, the series is $\sum n^2$, which also diverges by the Test for Divergence.

- We conclude that the interval of convergence is $1 < x < 5$.

8. (d) Choose the option that best describes the sequence

$$a_n = (-1)^n \frac{n}{n+2}, n \geq 1.$$

- The sequence is *bounded* since $|a_n| = \frac{n}{n+2} < \frac{n}{n} = 1$.

- It is *nonmonotonic* since its terms alternate in sign.

9. (a) Write down the Maclaurin series expansion for $\sin t$.

- We have $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}$ for $t \in \mathbb{R}$.

(b) Use part (a) to find the power series expansion for the sine integral $\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt$.

- Therefore,

$$\begin{aligned} \text{Si}(x) &= \int_0^x \frac{\sin t}{t} dt \\ &= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n} dt \\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)! (2n+1)} \right) \Big|_0^x \\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)! (2n+1)} \right) - \left(\sum_{n=0}^{\infty} 0 \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)! (2n+1)} \text{ for } x \in \mathbb{R}. \end{aligned}$$

10. Consider the series $\sum_{n=1}^{\infty} \frac{(-5)^{n+1}}{(2n+1)!}$.

(a) Prove that the series converges absolutely.

- The Ratio Test gives

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{5^{n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{5^{n+1}} = \frac{5}{(2n+2)(2n+3)} \rightarrow 0 < 1.$$

Hence the series converges absolutely.

(b) Use s_2 , the second partial sum, to approximate the sum of the series.

- We have $s_2 = \frac{25}{6} - \frac{125}{120} = \frac{500 - 125}{120} = \frac{375}{120} = \frac{75}{24} = \frac{25}{8} = 3\frac{1}{8}$.

(c) Find an upper bound on the remainder in using s_2 to approximate the sum of the series.

- The Alternating Series Estimation Theorem yields

$$|R_2| \leq |a_3| = \frac{5^4}{7!} = \frac{625}{5040} = \frac{125}{1008} \approx 0.124,$$

any one of which is fine.

11. Find the power series expansion about $a = 0$ for

$$f(x) = \frac{1}{1+9x^2} \text{ and determine its radius of convergence.}$$

- Use the Geometric Series Theorem!

$$\frac{1}{1 - (-9x^2)} = \sum_{n=0}^{\infty} (-9x^2)^n = \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n},$$

provided $|-9x^2| < 1$ or $|x| < \frac{1}{3}$.

- The radius of convergence is $R = \frac{1}{3}$.

12. The function $f(x) = \frac{x}{1+x^3}$ has a Taylor series expansion about $a = 1$ given by

$$f(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \dots$$

(a) Find these values: $f(1)$, $f'(1)$, and $f''(1)$.

- Compute the first two derivatives of f .

$$f(x) = \frac{x}{1+x^3}$$

$$f'(x) = \frac{(1+x^3)(1) - x(3x^2)}{(1+x^3)^2} = \frac{1-2x^3}{(1+x^3)^2}$$

$$f''(x) = \frac{(1+x^3)^2(-6x^2) - (1-2x^3) \cdot 2(1+x^3)(3x^2)}{(1+x^3)^4}$$

- Evaluate f and these derivatives at 1.

$$f(1) = \frac{1}{2}$$

$$f'(1) = -\frac{1}{4}$$

$$f''(1) = \frac{-24 + 12}{16} = -\frac{12}{16} = -\frac{3}{4}.$$

(b) Now compute these coefficients: a_0 , a_1 , and a_2 .

- We have $a_n = \frac{f^{(n)}(1)}{n!}$ for $n \geq 0$.

$$a_0 = \frac{1}{2}$$

$$a_1 = -\frac{1}{4}$$

$$a_2 = -\frac{3}{8}$$