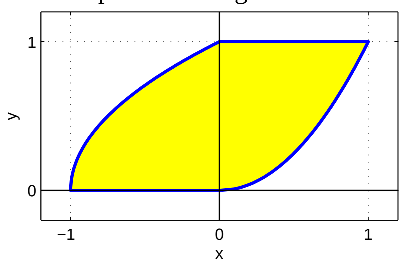
Spring 2008 Math 152
Exam 1A: Problems and Solutions
Mon, 18/Feb ©2008 Art Belmonte

1. (d) Find the area (in cm^2) of the region in the xy-plane bounded by the following curves.

$$x = y^2 - 1 \qquad x = \sqrt{y} \qquad y = 0 \qquad y = 1$$

- Here is a plot of the region.



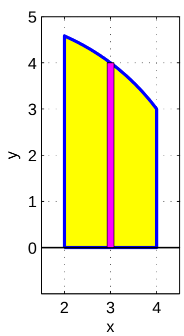
- The area is

$$\begin{aligned} A &= \int_0^1 y^{1/2} - \left(y^2 - 1\right)\,dy \\ &= \left(\tfrac{2}{3}y^{3/2} - \tfrac{1}{3}y^3 + y\right)\Big|_0^1 \\ &= \left(\tfrac{2}{3} - \tfrac{1}{3} + 1\right) - 0 = \tfrac{4}{3}\ \text{cm}^2. \end{aligned}$$

2. (b) Find the volume (in cm^3) of the solid obtained by rotating the region bounded by the given curves about the x-axis.

$$y = \sqrt{25 - x^2} \qquad y = 0 \qquad x = 2 \qquad x = 4$$

- Here is a plot of the region.



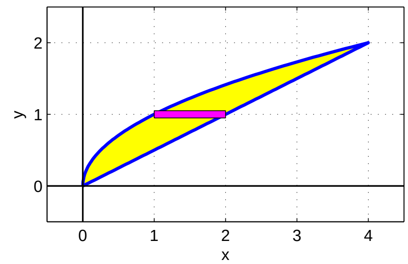
- The volume is

$$\begin{aligned} V &= \int \pi r^2\,dx = \int \pi y^2\,dx \\ &= \pi \int_2^4 25 - x^2\,dx \\ &= \pi\left(25x - \tfrac{1}{3}x^3\right)\Big|_2^4 \\ &= \pi\left(100 - \tfrac{64}{3}\right) - \pi\left(50 - \tfrac{8}{3}\right) \\ &= \left(50 - \tfrac{56}{3}\right)\pi = \tfrac{94}{3}\pi\ \text{cm}^3. \end{aligned}$$

3. (a) Find the volume (in cm^3) of the solid obtained by rotating the region in the first quadrant bounded by the given curves about the y-axis.

$$x = y^2 \qquad x = 2y$$

- When the curves intersect, their x-coordinates are equal. Thus $y^2 = 2y$ or $y\,(y - 2) = 0$ implies $y = 0, 2$. Here is a plot of the region.

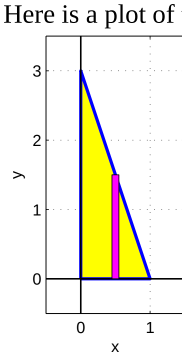


- Via "washers," the volume is

$$\begin{aligned} V &= \int \pi r_o^2 - \pi r_i^2\,dy \\ &= \pi \int_0^2 (2y)^2 - \left(y^2\right)^2\,dy \\ &= \pi \int_0^2 4y^2 - y^4\,dy \\ &= \pi\left(\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right)\Big|_0^2 \\ &= \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) - 0 \\ &= 32\pi\left(\tfrac{5-3}{15}\right) = \tfrac{64}{15}\pi\ \text{cm}^3. \end{aligned}$$

4. (e) Find the volume (in cm^3) of the solid S whose base in the xy-plane is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,3)$. Cross-sections perpendicular to the x-axis are squares.

- Here is a plot of the base.

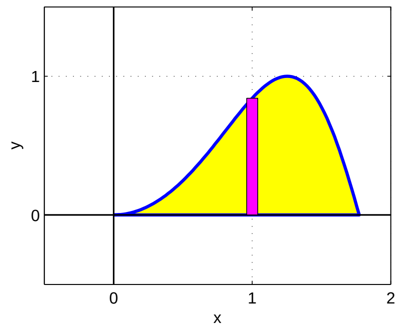


- The volume is

$$\begin{aligned} V &= \int y^2\,dx \\ &= \int_0^1 (3 - 3x)^2\,dx \\ &= \int_0^1 9 - 18x + 9x^2\,dx \\ &= \left(9x - 9x^2 + 3x^3\right)\Big|_0^1 \\ &= 3 - 0 = 3\ \text{cm}^3. \end{aligned}$$

5. (c) Use the method of cylindrical shells to find the volume (in cm^3) generated by rotating the region in the first quadrant bounded by the x-axis and the curve $y = \sin(x^2)$, $0 \le x \le \sqrt{\pi}$, about the y-axis.

- Here is a plot of the region.



- The volume is

$$\begin{aligned} V &= \int 2\pi r h \, dx \\ &= \pi \int_0^{\sqrt{\pi}} 2x \sin\left(x^2\right) \, dx \\ &= \left(-\pi \cos\left(x^2\right)\right)\Big|_0^{\sqrt{\pi}} \\ &= (\pi) - (-\pi) = 2\pi \ \text{cm}^3. \end{aligned}$$

[In the third step you may use the Substitution Rule.]

6. (b) A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to 15 cm. How much work (in joules) is done in stretching the spring from 12 cm to 20 cm?

- Via Hooke's Law, $30 = F = kx = k\left(\frac{15-12}{100}\right)$ implies $k = 1000$.
- The work is

$$\begin{aligned} W &= \int_0^{(20-12)/100} 1000x \, dx \\ &= 500x^2\Big|_0^{8/100} \\ &= \tfrac{320}{100} - 0 = \tfrac{16}{5} \text{ J}. \end{aligned}$$

7. (e) Find the average value of $f(x) = \dfrac{3}{(1+x)^2}$ on $[1, 3]$.

- The average value is

$$\begin{aligned} f_{ave} &= \frac{1}{3-1} \int_1^3 3(1+x)^{-2} \, dx \\ &= \left(-\frac{3/2}{1+x}\right)\Big|_1^3 \\ &= \left(-\tfrac{3}{8}\right) - \left(-\tfrac{3}{4}\right) = \tfrac{3}{8}. \end{aligned}$$

[In the second step you may use the Substitution Rule.]

8. (c) Evaluate $\displaystyle\int x^3 \ln x \, dx$.

- Use integration by parts. Let $u = \ln x$ and $dv = x^3 \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = \frac{1}{4}x^4$.
- Thus $\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx$ $= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$.

9. (d) Evaluate $\displaystyle\int_0^{\pi/2} \sin^5 x \cos^3 x \, dx$.

- This is a trigonometric integral. Use the trig identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned} & \int_0^{\pi/2} \sin^5 x \cos^2 x \cos x \, dx \\ =& \int_0^{\pi/2} \sin^5 x \left(1 - \sin^2 x\right) \cos x \, dx \\ =& \int_0^{\pi/2} \left(\sin^5 x - \sin^7 x\right) \cos x \, dx \\ =& \left(\tfrac{1}{6}\sin^6 x - \tfrac{1}{8}\sin^8 x\right)\Big|_0^{\pi/2} \\ =& \left(\tfrac{1}{6} - \tfrac{1}{8}\right) - (0) \\ =& \frac{4-3}{24} = \frac{1}{24} \end{aligned}$$

[In the fourth step you may use the Substitution Rule.]

10. Evaluate $\displaystyle\int_0^1 x^5 e^{x^3} \, dx$.

- First use integration by parts to find an antiderivative $F(x)$ of the integrand. Let $u = x^3$ and $dv = x^2 e^{x^3} \, dx$. Then $du = 3x^2 \, dx$ and $v = \frac{1}{3}e^{x^3}$. Therefore,

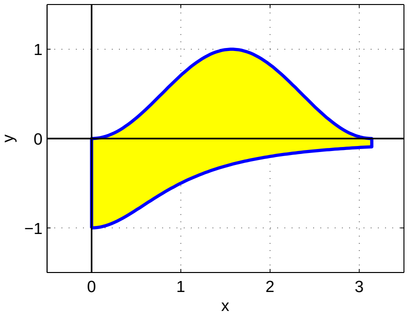
$$\begin{aligned} \int x^5 e^{x^3} \, dx &= \tfrac{1}{3}x^3 e^{x^3} - \int x^2 e^{x^3} \, dx \\ F(x) &= \tfrac{1}{3}e^{x^3}\left(x^3 - 1\right). \end{aligned}$$

- Hence $\int_0^1 x^5 e^{x^3} \, dx = F(x)|_0^1 = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$.

11. Find the area (in cm^2) of the region in the xy-plane bounded by the following curves.

$$y = \sin^2 x \qquad y = -\frac{1}{1+x^2} \qquad x = 0 \qquad x = \pi$$

- Here is a plot of the region.



- The area is

$$\begin{aligned} A &= \int_0^{\pi} \sin^2 x - \left(-\frac{1}{1+x^2}\right) dx \\ &= \int_0^{\pi} \frac{1-\cos 2x}{2} + \frac{1}{1+x^2}\, dx \\ &= \left(\frac{x - \frac{1}{2}\sin 2x}{2} + \tan^{-1} x\right)\Bigg|_0^{\pi} \\ &= \left(\tfrac{\pi}{2} + \tan^{-1}\pi\right) - 0 = \tfrac{\pi}{2} + \tan^{-1}\pi\,. \end{aligned}$$

[In the second step we used a trig identity.]

12. Evaluate $\displaystyle\int_{-3}^{3} \frac{27}{(36-x^2)^{3/2}}\, dx$.

- First off, the integrand is even and the interval of integration is symmetric about 0. Thus the original integral is equivalent to $2\displaystyle\int_0^3 \frac{27}{(36-x^2)^{3/2}}\, dx$. (This makes hand computation easier, although it is by no means necessary.)
- Use trigonometric substitution. Let $x = 6\sin\theta$. Then $dx = 6\cos\theta\, d\theta$.
- When $x = 0$, we have $\theta = 0$; whereas when $x = 3$, we have $\theta = \pi/6$. Therefore,

$$\begin{aligned} 2\int_0^{\pi/6} \frac{27 \cdot 6\cos\theta\, d\theta}{6^3\cos^3\theta} &= \frac{3}{2}\int_0^{\pi/6} \sec^2\theta\, d\theta \\ &= \tfrac{3}{2}\tan\theta\Big|_0^{\pi/6} \\ &= \left(\frac{3}{2}\right)\left(\frac{1/2}{\sqrt{3}/2}\right) - 0 \\ &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Here we recalled that $\tan\theta = \sin\theta/\cos\theta$.

13. Evaluate $\displaystyle\int \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2}\, dx$.

- The degree of the numerator is not less than that of the denominator. So first divide the numerator by the denominator to obtain $\displaystyle\int 1 + \frac{-4}{x^3 - 2x^2}\, dx = \star$.
- Use the method of partial fractions on the second term.

$$\begin{aligned} \frac{-4}{x^2(x-2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \\ -4 &= Ax(x-2) + B(x-2) + Cx^2 \\ -4 &= (A+C)x^2 + (-2A+B)x - 2B \end{aligned}$$

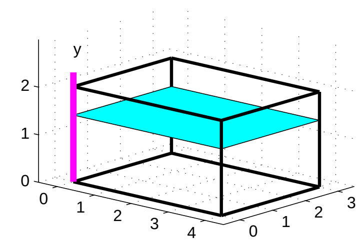
Equate like coefficients: $-2B = -4$ implies $B = 2$. Next, $-2A + B = 0$ implies $A = \frac{1}{2}B = 1$. Finally, $A + C = 0$ implies $C = -A = -1$.

- Accordingly,

$$\begin{aligned} \star &= \int 1 + \frac{1}{x} + 2x^{-2} - \frac{1}{x-2}\, dx \\ &= x + \ln|x| - \frac{2}{x} - \ln|x-2| + C. \end{aligned}$$

14. An aquarium 4 m long, 3 m wide, and 2 m deep is filled with water. Find the amount of work (in joules) required to pump all the water out of the aquarium. The mass density of water is $\rho = 1000\ \text{kg/m}^3$ and the acceleration due to gravity is $g = 9.8 = \frac{98}{10}\ \text{m/s}^2$. The answer is an *integer*. Yes, Citizen, simplify it *all the way!*

- Here is a diagram of the aquarium together with a typical differential layer of water within.



- Herewith the so-called "march of the differentials." It shows the differential volume, mass, force (weight), as well as work required to lift the layer to the top of the aquarium.

$$\begin{aligned} dV &= (4)(3)\,dy = 12\,dy \\ dm &= \rho\, dV = 12\rho\, dy \\ dF &= (dm)\,g = 12\rho g\, dy \\ dW &= (dF)(D) = (dF)(2-y) = 12\rho g\,(2-y)\ dy \end{aligned}$$

- Integrate to find the total work done.

$$\begin{aligned} W = \int dW &= \int_0^2 12\,(1000)\,(98/10)\,(2-y)\ dy \\ &= 1200\,(100-2)\int_0^2 2 - y\, dy \\ &= (120000 - 2400)\left(2y - \tfrac{1}{2}y^2\right)\Big|_0^2 \\ &= (120000 - 2400)(4-2) - 0 \\ &= 240000 - 4800 = 235{,}200\ \text{J} \end{aligned}$$