# Spring 2008 Math 152
# Exam 3B: Solutions
# Fri, 25/Apr ©2008 Art Belmonte

1. (a) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\left(1+(\ln n)^2\right)}$ converges or diverges and why.

   - The series converges by the Integral Test.

$$
\begin{aligned}
&\int_1^{\infty} \frac{1}{1+(\ln x)^2}\,\frac{1}{x}dx \\
=& \lim_{b\to\infty}\int_1^{b} \frac{1}{1+(\ln x)^2}\,\frac{1}{x}dx \\
=& \lim_{b\to\infty} \tan^{-1}(\ln x)\Big|_1^b \\
=& \lim_{b\to\infty}\left(\tan^{-1}(\ln b) - 0\right) = \frac{\pi}{2}
\end{aligned}
$$

2. (c) Determine whether the series $\displaystyle\sum_{n=6}^{\infty} \frac{3+\pi^n}{e^n - n - 64}$ converges or diverges and why.

   - Observe that $\dfrac{3+\pi^n}{e^n-n-64} \ge \dfrac{\pi^n}{e^n} = \left(\dfrac{\pi}{e}\right)^n$. Accordingly, our series diverges by the Comparison Test since $\sum(\pi/e)^n$ diverges by the Geometric Series Theorem because $|r| = \pi/e > 1$.

3. (c) Choose the option that best describes the triangle in space whose vertices are $P\,(2,-1,0)$, $Q\,(4,1,1)$, $R\,(4,-5,4)$.

   - Compute the lengths of the triangle's sides.

$$
\begin{aligned}
\left\|\overrightarrow{PQ}\right\| &= \sqrt{4+4+1} = \sqrt{9} \\
\left\|\overrightarrow{QR}\right\| &= \sqrt{0+36+9} = \sqrt{45} \\
\left\|\overrightarrow{RP}\right\| &= \sqrt{4+16+16} = \sqrt{36}
\end{aligned}
$$

   - Since $9+36=45$, this is a right triangle by the Pythagorean Theorem. But it is not isosceles since no two lengths are equal.

4. (a) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is divergent, absolutely convergent or just convergent and why.

   - This series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Theorem since $b_n = |a_n| = \frac{1}{\sqrt{n}} \downarrow 0$. (That is, the $b_n$ form a decreasing sequence whose limit is zero.)
   - However, $\sum |a_n| = \sum \frac{1}{\sqrt{n}}$ is a divergent $p$-series since $p = \frac{1}{2} \le 1$.
   - Accordingly, our series is converges, but not absolutely. It is said to be *conditionally convergent*.

5. (e) Let $\mathbf{a} = \langle 1, 0, -2\rangle$ and $\mathbf{b} = \langle -1, -2, -4\rangle$. Compute the angle $\theta$ (in radians) between $\mathbf{a}$ and $\mathbf{b}$.

   - The angle between $\mathbf{a}$ and $\mathbf{b}$ is

$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right) \\
&= \cos^{-1}\left(\frac{-1+0+8}{\sqrt{5}\sqrt{21}}\right) \\
&= \cos^{-1}\left(\frac{\sqrt{105}}{15}\right).
\end{aligned}
$$

6. (b) Determine whether $\displaystyle\sum_{n=0}^{\infty}\left(7\left(\frac{1}{5}\right)^n - 4\left(-\frac{1}{9}\right)^n\right)$ converges or diverges. If it converges, find its sum.

   - Via the Geometric Series Theorem and the sum laws for series, this series converges to

$$
\begin{aligned}
\frac{7}{1-\frac{1}{5}} - \frac{4}{1+\frac{1}{9}} &= \frac{35}{5-1} - \frac{36}{9+1} \\
&= \frac{35}{4} - \frac{36}{10} \\
&= \frac{175-72}{20} = \frac{103}{20}.
\end{aligned}
$$

7. (c) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+5}}$ converges or diverges and why.

   - As $n \to \infty$,

$$
|a_n| = \frac{n}{\sqrt{n^2+5}} = \frac{1}{\sqrt{1+\frac{5}{n^2}}} = 1.
$$

   The series diverges by the Test for Divergence.

8. (b) Find the limit of the sequence $a_n = \dfrac{\ln n}{\sqrt{n}}$.

   - Apply L'Hospital's Rule.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\ln n}{n^{1/2}} &\overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1/n}{\frac{1}{2}n^{-1/2}} \\
&= \lim_{n\to\infty} \frac{1}{\frac{1}{2}n^{1/2}} = 0
\end{aligned}
$$

9. (d) Find the third degree Taylor polynomial of $f(x) = e^x$ at $a = \ln 3$.

- Now $f^{(n)}(x) = e^x$, whence $f^{(n)}(\ln 3) = 3$ for $n \geq 0$.
- The desired Taylor polynomial is therefore

$$\begin{aligned} & \sum_{n=0}^{3} \frac{f^{(n)}(\ln 3)}{n!}(x-\ln 3)^n \\ = & \sum_{n=0}^{3} \frac{3}{n!}(x-\ln 3)^n. \end{aligned}$$

10. (b) Determine whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,(n+1)!}$ is divergent, absolutely convergent, or just convergent and why.

- The series diverges by Ratio Test.

$$\begin{aligned} & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ = & \lim_{n\to\infty}\frac{(2n+2)!}{(n+1)!\,(n+2)!}\,\frac{n!\,(n+1)!}{(2n)!} \\ = & \lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)(n+2)} \\ = & \lim_{n\to\infty}\left(\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right)\left(\frac{2+\frac{2}{n}}{1+\frac{2}{n}}\right) = 4 > 1 \end{aligned}$$

11. (d) Determine whether the series $\sum_{n=5}^{\infty} \frac{1}{(n+1)(n+2)}$ converges or diverges. If it converges, find its sum.

- Split the $n^{\text{th}}$ term into partial fractions.

$$\begin{aligned} \frac{1}{(n+1)(n+2)} &= \frac{A}{n+1}+\frac{B}{n+2} \\ 1 &= A(n+2)+B(n+1) \\ 1 &= (A+B)\,n+(2A+B) \end{aligned}$$

  Equating like coefficients, we have $A+B=0$ and $2A+B=1$. The first equation yields $B=-A$. We substitute this result into the second equation to obtain $A=1$, whence $B=-1$.

- The $m^{\text{th}}$ partial sum of this telescoping series $\sum_{n=5}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$ is $s_m = \frac{1}{6}-\frac{1}{m+2}$. The series converges to $\lim_{m\to\infty} s_m = \frac{1}{6}$.

12. Compute the Taylor series for $g(x) = \ln(x+4)$ about $x = a = -3$.

- Repeated differentiation of $g$ reveals the pattern $g^{(n)}(x) = (-1)^{n-1}(n-1)!\,(x+4)^{-n}$, whence $g^{(n)}(-3) = (-1)^{n-1}(n-1)!$ for $n \geq 1$.
- The Taylor series for $g$ at $a = 3$ is therefore

$$\begin{aligned} & \sum_{n=0}^{\infty}\frac{g^{(n)}(-3)}{n!}(x-(-3))^n \\ = & \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x+3)^n \end{aligned}$$

  since $g(-3) = \ln 1 = 0$.

13. Find the radius $R$ and interval $I$ of convergence of the power series $\sum_{n=0}^{\infty}\frac{nx^n}{4^n\left(n^2+1\right)}$.

- For convergence, the Ratio Test requires

$$\begin{aligned} & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ = & \lim_{n\to\infty}\frac{(n+1)\,|x|^{n+1}}{4^{n+1}\left(n^2+2n+2\right)}\,\frac{4^n\left(n^2+1\right)}{n\,|x|^n} \\ = & \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}\right)\frac{|x|}{4} \\ = & \frac{|x|}{4} < 1 \end{aligned}$$

  or $|x| < 4$. The radius of convergence is $R = 4$.

- [The Root Test yields the same result more quickly.] $\sqrt[n]{|a_n|} = \frac{\sqrt[n]{n}\,|x|}{4\sqrt[n]{n^2+1}} \to \frac{|x|}{4} < 1$ or $|x| < 4$.
- For $x = -4$, the series is $\sum \frac{(-1)^n\,n}{n^2+1}$. It converges by the Alternating Series Test since $|a_n| = \frac{1}{n+\frac{1}{n}} \downarrow 0$.
- For $x = 4$, the series is $\sum a_n = \sum \frac{n}{n^2+1}$. Let $b_n = \frac{1}{n}$. Note that $\frac{a_n}{b_n} = \frac{n^2}{n^2+1} = \frac{1}{1+\frac{1}{n^2}} \to 1$ as $n \to \infty$. Since $\sum \frac{1}{n}$ diverges (it's a $p$-series with $p = 1 \leq 1$), we conclude that $\sum a_n$ diverges by the Limit Comparison Test.
- Therefore, the interval of convergence is $I = [-4, 4)$.

14. Use series to approximate the integral $\int_0^{1/2} e^{-x^3}\,dx$ to within $\epsilon = 10^{-2}$. Use the fewest number of terms possible to attain the prescribed accuracy. Express your final answer as a fraction.

- Recall the Maclaurin series $e^z = 1 + z + \frac{z^2}{2!} + \cdots$, which converges for all real $z$.
- Hence $e^{-x^3} = 1 - x^3 + \frac{x^6}{2} - \cdots$ for all real $x$.

- We may therefore integrate term-by-term to obtain

$$\begin{aligned}\int_0^{1/2} e^{-x^3}\,dx &= \int_0^{1/2} 1 - x^3 + \frac{x^6}{2} - \cdots\,dx \\ &= \left(x - \frac{x^4}{4} + \frac{x^7}{2\,(7)} - \cdots\right)\Bigg|_0^{1/2} \\ &= \left(\frac{1}{2} - \frac{1}{64} + \frac{1}{7\,(256)} - \cdots\right) - 0.\end{aligned}$$

- By the Alternating Series Estimation Theorem, we have

$$|R_2| \le |a_3| = \frac{1}{7\,(256)} < 10^{-2}.$$

Therefore $\int_0^{1/2} e^{-x^3}\,dx \approx s_2 = \frac{1}{2} - \frac{1}{64} = \frac{31}{64}$.

15. (a) Find a power series for $f(x) = \frac{3}{7-5x}$ centered at $a = 0$.

    - Via the Geometric Series Theorem, a power series representation for $g(x) = \frac{3}{7-5x}$ is

    $$\begin{aligned}\frac{3/7}{1-\frac{5}{7}x} &= \frac{3}{7}\sum_{n=0}^{\infty}\left(\frac{5}{7}x\right)^n \\ &= \sum_{n=0}^{\infty}\frac{3}{7}\left(\frac{5}{7}\right)^n x^n,\end{aligned}$$

    provided $|r| = \left|\frac{5}{7}x\right| < 1$ or $|x| < \frac{7}{5}$. The radius of convergence of this series for $f$ is $R = \frac{7}{5}$.

    (b) Use the result from part (a) to determine a power series representation for $f'(x)$ centered at $a = 0$. Also give its radius of convergence.

    - On $\left(-\frac{7}{5}, \frac{7}{5}\right)$, the power series for $f$ is differentiable and

    $$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{3}{7}\left(\frac{5}{7}\right)^n x^n\right) = \sum_{n=1}^{\infty}\frac{3}{7}\left(\frac{5}{7}\right)^n n x^{n-1}.$$

    Moreover, this power series representation for $f'$ has the same radius of convergence as that of $f$; that is, $R = \frac{7}{5}$.