Math 152 Spring 2009 Exam I Solutions-Form A

1. d: $f_{ave} = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$. Using u-substitution, with $u = x^2$, we obtain
$f_{ave} = \frac{1}{2\sqrt{\pi}}\int_0^{\pi} \sin(u)\,du = -\frac{1}{2\sqrt{\pi}}\cos u|_0^{\pi} = \frac{1}{\sqrt{\pi}}$

2. a: Use integration by parts, with $u = \ln x$ and $dv = x^3\,dx$. Then $du = \frac{dx}{x}$ and $v = \frac{x^4}{4}$. Thus $\int_1^2 x^3\ln x\,dx = \frac{x^4}{4}\ln x|_1^2 - \int_1^2 \frac{x^4}{4}\frac{dx}{x}\,dx = \left(\frac{x^4}{4}\ln x - \frac{x^4}{16}\right)|_1^2 = 4\ln 2 - \frac{15}{16}$

3. d: The total work is the work done to pull the first 10 feet of the rope to the top plus the work done to pull the remaining weight of rope up 10 feet. The rope weighs $\frac{1}{2}$ pounds per foot, hence $W = \int_0^{10} \frac{1}{2}x\,dx + 40(10)(\frac{1}{2}) = 225$ foot pounds.

4. b: Here, we will use the identity
$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$. Thus
$\int \cos^2(2x)\,dx = \int \frac{1}{2}(1+\cos(4x))\,dx = \frac{1}{2}\left(x + \frac{1}{4}\sin(4x)\right) + C$

5. e: Using the method of cylindrical shells, $V = \int_0^2 2\pi x(2x - x^2)\,dx = \frac{8\pi}{3}$. Using the method of washers, $V = \int_0^4 \pi(y - \frac{y^2}{4})\,dy = \frac{8\pi}{3}$.

6. b: The curves $y = x^2$ and $y = 8 - x^2$ intersect at the points $(2, 4)$ and $(-2, 4)$. Thus
$A = \int_{-2}^{2}(8 - x^2 - x^2)\,dx = \frac{64}{3}$.

7. b: $f_{ave} = \frac{1}{b}\int_0^b (3x^2 - 2x)\,dx = \frac{1}{b}(x^3 - x^2)|_0^b = \frac{1}{b}(b^3 - b^2) = b^2 - b$. Now solving $f_{ave} = 2$ we obtain $b^2 - b = 2$, thus $b^2 - b - 2 = 0$ yielding a positive value of $b = 2$.

8. b: Using the method of cylindrical shells, $V = \int_1^5 \left(2\pi x\frac{1}{x}\right)\,dx = 8\pi$.

9. c: $\int_0^{\pi/4} \tan^4 x\sec^4 x\,dx =$
$= \int_0^{\pi/4} \tan^4 x\sec^2 x\sec^2 x\,dx$
$= \int_0^{\pi/4} \tan^4 x(\tan^2 x + 1)\sec^2 x\,dx$. Now, let $u = \tan x$. Then $du = \sec^2 x\,dx$. Also note if $x = 0$, $u = 0$ and if $x = \frac{\pi}{4}$, $u = 1$. Thus $\int_0^{\pi/4} \tan^4 x(\tan^2 x + 1)\sec^2 x\,dx = \int_0^1 u^4(u^2+1)\,du = \frac{12}{35}$

10. a: Use integration by parts, where $u = x$ and $dv = \sin(\pi x)\,dx$. Then $du = dx$ and $v = -\frac{1}{\pi}\cos(\pi x)$. Thus $\int_0^1 x\sin(\pi x)\,dx = \frac{-x}{\pi}\cos(\pi x)|_0^1 - \int_0^1 \frac{-1}{\pi}\cos(\pi x)\,dx = \left(-\frac{x}{\pi}\cos(\pi x) + \frac{1}{\pi^2}\sin(\pi x)\right)|_0^1 = \frac{1}{\pi}$

11. The curves $y = x^2 + 1$ and $y = 2$ intersect at $x = \pm 1$ Using the washer method with outer radius $R = 3 - (x^2 + 1) = 2 - x^2$ and inner radius $r = 1$, we get $V = \int_{-1}^{1} \pi\left(R^2 - r^2\right)\,dx = \int_{-1}^{1} \pi\left((2-x^2)^2 - (1)^2\right)\,dx = \frac{56\pi}{15}$

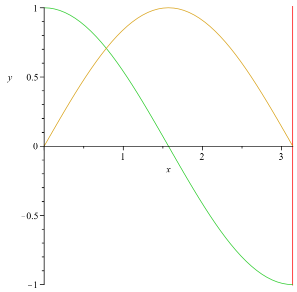
12. Use trigonometric substitution with $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$. Thus $\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta}{9\sin^2\theta\sqrt{9 - 9\sin^2\theta}}\,d\theta = \int \frac{3\cos\theta}{9\sin^2\theta\sqrt{9\cos^2\theta}}\,d\theta = \int \frac{d\theta}{9\sin^2\theta}\,d\theta = \frac{1}{9}\int \csc^2\theta\,d\theta = -\frac{1}{9}\cot\theta + C$.
Now since $x = 3\sin\theta$, $\cot\theta = \frac{\sqrt{9-x^2}}{x}$, therefore $-\frac{1}{9}\cot\theta + C = -\frac{\sqrt{9-x^2}}{9x} + C$

13. A cross section of water is volume
$\frac{2}{3}(3-y)(8)dy$, thus the work done in lifting this section of water up to the top of the tank is $9800\frac{16}{3}(3-y)y\,dy$. Therefore to empty the tank, $W = \int_0^3 9800\frac{16}{3}(3-y)y\,dy = 235200$ Joules.

14. Since the cross sections are squares, the volume of a typical cross section perpendicular to the x-axis is $V_i = s^2dy$ where s is the side of the square. Now,

since the cross sections are perpendicular to the x-axis, $s = y$. Finding the equation of the line gives us $y = -2x + 2$. Therefore $V = \int_0^1 (-2x+2)^2\,dx = \frac{4}{3}$

15. The curves $y = \sin x$ and $y = \cos x$ intersect at $x = \frac{\pi}{4}$ for $0 \le x \le \pi$.



The area must be split into 2 separate integrals.
$A = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\,dx$

$A = (\sin x + \cos x)|_0^{\pi/4} + (-\cos x - \sin x)|_{\pi/4}^{\pi} = 2\sqrt{2}$