

## Math 152 Spring 2009 Exam II Solutions-Form A

1. d: The length of the curve  $y = \sqrt{x^3}$  from  $x = 0$  to  $x = 4$  is  $L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ . Now,  $y = x^{3/2}$ , therefore  $\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$ .  $L = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx$ . Using  $u$ -substitution, with  $u = 1 + \frac{9}{4}x$ , we obtain

$$L = \frac{4}{9} \int_1^{10} \sqrt{u} du = \frac{8}{27} (10\sqrt{10} - 1).$$

2. b: This integral is improper at  $\infty$ , thus  $\int_0^\infty xe^{-3x} dx = \lim_{t \rightarrow \infty} \int_0^t xe^{-3x} dx$ . Using integration by parts with  $u = x$  and  $dv = e^{-3x} dx$ :  $\lim_{t \rightarrow \infty} \int_0^t xe^{-3x} dx = \lim_{t \rightarrow \infty} \left(-\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\right) \Big|_0^t$   
 $= \lim_{t \rightarrow \infty} \left(-\frac{t}{3e^{3t}} - \frac{1}{9e^{3t}} + \frac{1}{9}\right)$   
 $= \lim_{t \rightarrow \infty} \left(-\frac{t}{3e^{3t}}\right) - \lim_{t \rightarrow \infty} \left(\frac{1}{9e^{3t}}\right) + \lim_{t \rightarrow \infty} \left(\frac{1}{9}\right)$   
 $= 0 + 0 + \frac{1}{9}$ . Note:  $\lim_{t \rightarrow \infty} \left(-\frac{t}{3e^{3t}}\right)$  was found to be 0 by L'Hospital's rule.

3. a: To find  $\int_1^2 \frac{x^2 + 1}{x^2 + x} dx$ , we must first find the partial fraction decomposition of  $\frac{x^2 + 1}{x^2 + x}$ . Since the power in the denominator is not higher than the numerator, we must use long division. Doing this, we find  $\frac{x^2 + 1}{x^2 + x} = 1 + \frac{-x + 1}{x^2 + x}$ . Now,  $\frac{-x + 1}{x^2 + x} = \frac{A}{x} + \frac{B}{x + 1}$ . Solving for  $A$  and  $B$ , we find that  $A = 1$  and  $B = -2$ . Hence  $\int_1^2 \frac{x^2 + 1}{x^2 + x} dx = \int_1^2 \left(1 + \frac{1}{x} - \frac{2}{x + 1}\right) dx$   
 $= (x + \ln|x| - 2 \ln|x + 1|) \Big|_1^2$   
 $= 1 + 3 \ln 2 - 2 \ln 3$ .

4. e:  $A = \int_0^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ , where  $\frac{dy}{dx} = -2x$ .

$$A = \int_0^2 2\pi x \sqrt{1 + (-2x)^2} dx. \quad \text{Using } u\text{-substitution with } u = 1 + 4x^2, \text{ we find}$$

$$A = \int_0^2 2\pi x \sqrt{1 + 4x^2} dx = \int_1^{17} \frac{2\pi}{8} \sqrt{u} du$$

$$= \frac{\pi}{6} (17\sqrt{17} - 1).$$

5. a:  $|E_T| \leq \frac{K(b-a)^3}{12n^2}$ , where  $a = 1$ ,  $b = 3$ , and  $K = \max|f''(x)|$  for  $1 \leq x \leq 3 = \max\left|\frac{-1}{x^2}\right|$  for  $1 \leq x \leq 3$ , hence  $K = 1$ . This yields  $|E_T| \leq \frac{1(2)^3}{12n^2} \leq \frac{1}{2400}$ . This yields  $1600 \leq n^2$ , hence  $n$  must be at least 40.

6. e: Since the pool was measured at 2 meter intervals, we know  $\Delta x = 2$ . The partition points are  $x_0, x_1, x_2, x_3$  and  $x_4$  where  $f(x_0) = 0$ ,  $f(x_1) = 1.5$ ,  $f(x_2) = 2$ ,  $f(x_3) = 3$  and  $f(x_4) = 0$ . Now

$$S_4 = \frac{2}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4))$$

$$= \frac{2}{3} (0 + 4(1.5) + 2(2) + 4(3) + 0) = \frac{44}{3}$$

7. b:  $\int_1^\infty \frac{dx}{x + e^{5x}} < \int_1^\infty \frac{dx}{e^{5x}}$ . Now

$$\int_1^\infty \frac{dx}{e^{5x}} = \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{e^{5x}} = \lim_{t \rightarrow \infty} \frac{1}{5e^{5x}} \Big|_1^t = \frac{1}{5e^5}$$

Thus by the comparison theorem, since the larger integral converges so does the smaller. Hence  $\int_1^\infty \frac{dx}{x + e^{5x}}$  converges by comparison to  $\int_1^\infty \frac{dx}{e^{5x}}$ .

8. a: First we will separate the differential equation:

$$\frac{du}{dt} = e^{2t-u} \text{ is equivalent to } e^u du = e^{2t} dt. \text{ Integrate both sides: } e^u = \frac{1}{2}e^{2t} + C. \text{ Now, } u(0) = 1 \text{ gives } e = \frac{1}{2} + C, \text{ thus } C = e - \frac{1}{2}. \text{ Substitute this back in gives } e^u = \frac{1}{2}e^{2t} + e - \frac{1}{2}, \text{ thus } u = \ln\left(\frac{1}{2}e^{2t} + e - \frac{1}{2}\right), \text{ thus } u(1) = \ln\left(\frac{1}{2}e^2 + e - \frac{1}{2}\right)$$

9. c:  $x = \sin t, y = \cos t$ , thus

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = -\sin t. \text{ Hence}$$

$$L = \int_0^{\pi/3} 2\pi \cos t \sqrt{\cos^2 t + \sin^2 t} dt$$

$$= \int_0^{\pi/3} 2\pi \cos t dt = \sqrt{3}\pi$$

10. c:  $x = e^{8y}$ , thus  $\frac{dx}{dy} = 8e^{8y}$ .

$$SA = \int_0^1 2\pi y \sqrt{1 + 64e^{16y}} dy.$$

11.  $F = \int \rho g dA$ . We orient the semicircular plate so the origin passes through the center of the diameter. Thus the depth  $d$  of an arbitrary strip of oil is  $d = y$  and the area is  $A = 2xdy$  where  $x^2 + y^2 = 1$ . Hence  $A = 2\sqrt{1-y^2}dy$ . This yields  $F = \int_0^1 2\rho g y \sqrt{1-y^2} dy$ . Integrate with  $u$  substitution with  $u = 1 - y^2$  we obtain  $F = \frac{2}{3}\rho g = 6000N$ .

12. Let  $y = y(t)$  be the amount of salt in the tank at time  $t$ . Since we are starting with pure water,  $y(0) = 0$ . Now,

$$\frac{dy}{dt} = \left(0.01 \frac{kg}{L}\right) \left(20 \frac{L}{m}\right) - \left(\frac{y}{250} \frac{kg}{L}\right) \left(20 \frac{L}{m}\right).$$

Thus  $\frac{dy}{dt} = 0.2 \frac{kg}{m} - \frac{2}{25} y \frac{kg}{m}$ . We can solve this equation as linear or separable. I will solve it as separable:  $\frac{dy}{dt} = \frac{5-2y}{25}$ , thus  $\frac{dy}{5-2y} = \frac{1}{25} dt$ . Integrating both sides yields

$$-\frac{1}{2} \ln(5-2y) = \frac{1}{25} t + C. \text{ Since } y(0) = 0, \text{ this gives } C = -\frac{1}{2} \ln(5). \text{ } -\frac{1}{2} \ln(5-2y) = \frac{1}{25} t - \frac{1}{2} \ln(5). \text{ Thus } \ln(5-2y) = -\frac{2}{25} t + \ln(5). \text{ Solve for } y:$$

$$5-2y = e^{-2t/25 + \ln 5} = 5e^{-2t/25}.$$

$$\text{Hence } y = \frac{5-5e^{-2t/25}}{2} \text{ kg of salt.}$$

13.  $\frac{x+2}{x^2(x^2+1)}$  has a partial fraction decomposition of

$$\text{the form } \frac{x+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}. \text{ Thus}$$

$$x+2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2.$$

$x+2 = (A+C)x^3 + (B+D)x^2 + Ax + B$ . Equate coefficients:  $A+C=0$ ,  $B+D=0$ ,  $A=1$  and  $B=2$ , which gives  $C=-1$  and  $D=-2$ . Thus

$$\int \frac{x+2}{x^2(x^2+1)} dx = \int \left( \frac{1}{x} + \frac{2}{x^2} + \frac{-x-2}{x^2+1} \right) dx$$

$$= \int \left( \frac{1}{x} + \frac{2}{x^2} + \frac{-x}{x^2+1} - \frac{2}{x^2+1} \right) dx$$

$$= \ln|x| - \frac{2}{x} - \frac{1}{2} \ln(x^2+1) - 2 \arctan x + C.$$

14.  $x \frac{dy}{dx} = x(\ln x)^2 + y$  is a linear differential equation. Getting this in linear form gives

$$\frac{dy}{dx} - \frac{1}{x} y = (\ln x)^2. \text{ Now the integrating factor is}$$

$$I(x) = e^{\int -1/x dx} = e^{-\ln x} = \frac{1}{x}.$$

$$\text{Hence } \frac{1}{x} \left( \frac{dy}{dx} - \frac{1}{x} y \right) = \frac{1}{x} (\ln x)^2.$$

Thus  $\frac{d}{dx} \left( y \frac{1}{x} \right) = \frac{1}{x} (\ln x)^2$ . Integrate both sides:

$$\int \frac{d}{dx} \left( y \frac{1}{x} \right) dx = \int \frac{1}{x} (\ln x)^2 dx. \text{ Using a } u\text{-substitution with } u = \ln x, \text{ we obtain}$$

$$y \frac{1}{x} = \frac{(\ln x)^3}{3} + C, \text{ thus } y = x \left( \frac{(\ln x)^3}{3} + C \right).$$

15. The curves  $y = \sqrt{x}$  and  $y = x^3$  intersect at  $x = 0$  and  $x = 1$ . Hence

$$A = \int_0^1 (\sqrt{x} - x^3) dx = \frac{5}{12}.$$

$$\bar{x} = \frac{1}{A} \int_0^1 x(\sqrt{x} - x^3) dx = \frac{12}{5} \int_0^1 (x^{3/2} - x^4) dx = \frac{12}{25}.$$

$$\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2} ((\sqrt{x})^2 - (x^3)^2) dx = \frac{12}{10} \int_0^1 (x - x^6) dx = \frac{3}{7}.$$

Thus the centroid is  $\left( \frac{12}{25}, \frac{3}{7} \right)$