Math 152 Spring 2009 Exam II Solutions-Form B

1. d: $A = \int_0^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, where $\frac{dy}{dx} = -2x$.
   $A = \int_0^2 2\pi x\sqrt{1+(-2x)^2}\,dx$. Using u-substitution with $u = 1 + 4x^2$, we find $A = \int_0^2 2\pi x\sqrt{1+4x^2}\,dx = \int_1^{17} \frac{2\pi}{8}\sqrt{u}\,du$
   $= \frac{\pi}{6}(17\sqrt{17}-1)$.

2. b: $|E_T| \le \frac{K(b-a)^3}{12n^2}$, where $a = 1$, $b = 3$, and $K = \max|f''(x)|$ for $1 \le x \le 3 = \max|\frac{-1}{x^2}|$ for $1 \le x \le 3$, hence $K = 1$. This yields $|E_T| \le \frac{1(2)^3}{12n^2} \le \frac{1}{2400}$. This yields $1600 \le n^2$, hence n must be atleast 40.

3. d: Since the pool was measured at 2 meter intervals, we know $\Delta x = 2$. The partition points are x_0, x_1, x_2, x_3 and x_4 where $f(x_0) = 0$, $f(x_1) = 1.5$,
   $f(x_2) = 2$, $f(x_3) = 3$ and $f(x_4) = 0$. Now
   $S_4 = \frac{2}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$
   $= \frac{2}{3}\left(0 + 4(1.5) + 2(2) + 4(3) + 0\right) = \frac{44}{3}$

4. e: The length of the curve $y = \sqrt{x^3}$ from $x = 0$ to $x = 4$ is $L = \int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. Now, $y = x^{3/2}$, therefore $\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$. $L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$. Using u-substitution, with $u = 1 + \frac{9}{4}x$, we obtain
   $L = \frac{4}{9}\int_1^{10} \sqrt{u}\,du = \frac{8}{27}\left(10\sqrt{10}-1\right)$.

5. b: First we will separate the differential equation:
   $\frac{du}{dt} = e^{2t-u}$ is equivalent to $e^u\,du = e^{2t}\,dt$. Integrate both sides: $e^u = \frac{1}{2}e^{2t} + C$. Now, $u(0) = 1$ gives $e = \frac{1}{2} + C$, thus $C = e - \frac{1}{2}$. Substitute this back in gives $e^u = \frac{1}{2}e^{2t} + e - \frac{1}{2}$, thus $u = \ln(\frac{1}{2}e^{2t} + e - \frac{1}{2})$, thus $u(1) = \ln(\frac{1}{2}e^2 + e - \frac{1}{2})$

6. a: $x = \sin t$, $y = \cos t$, thus
   $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = -\sin t$. Hence
   $L = \int_0^{\pi/3} 2\pi\cos t\sqrt{\cos^2 t + \sin^2 t}\,dt$
   $= \int_0^{\pi/3} 2\pi\cos t\,dt = \sqrt{3}\pi$

7. d: This integral is improper at ∞, thus $\int_0^\infty xe^{-3x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-3x}\,dx$. Using integration by parts with $u = x$ and $dv = e^{-3x}\,dx$:
   $\lim_{t\to\infty}\int_0^t xe^{-3x}\,dx = \lim_{t\to\infty}\left(-\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\right)\Big|_0^t$
   $= \lim_{t\to\infty}\left(-\frac{t}{3e^{3t}} - \frac{1}{9e^{3t}} + \frac{1}{9}\right)$
   $= \lim_{t\to\infty}\left(-\frac{t}{3e^{3t}}\right) - \lim_{t\to\infty}\left(\frac{1}{9e^{3t}}\right) + \lim_{t\to\infty}\left(\frac{1}{9}\right)$
   $= 0 + 0 + \frac{1}{9}$. Note: $= \lim_{t\to\infty}\left(-\frac{t}{3e^{3t}}\right)$ was found to be 0 by L'Hospital's rule.

8. a: $x = e^{8y}$, thus $\frac{dx}{dy} = 8e^{8y}$.
   $SA = \int_0^1 2\pi y\sqrt{1+64e^{16y}}\,dy.$

9. d: To find $\int_1^2 \frac{x^2+1}{x^2+x}\,dx$, we must first find the partial fraction decomposition of $\frac{x^2+1}{x^2+x}$. Since the power in the denominator is not higher than the numerator, we must use long division. Doing this, we find $\frac{x^2+1}{x^2+x} = 1 + \frac{-x+1}{x^2+x}$. Now, $\frac{-x+1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$. Solving for A and B, we find that $A = 1$ and $B = -2$. Hence $\int_1^2 \frac{x^2+1}{x^2+x}\,dx = \int_1^2 \left(1 + \frac{1}{x} - \frac{2}{x+1}\right)\,dx$
   $= (x + \ln|x| - 2\ln|x+1|)\,|_1^2$
   $= 1 + 3\ln 2 - 2\ln 3.$

10. c: $\int_1^\infty \frac{dx}{x+e^{5x}} < \int_1^\infty \frac{dx}{e^{5x}}$. Now $\int_1^\infty \frac{dx}{e^{5x}} = \lim_{t\to\infty}\int_1^t \frac{dx}{e^{5x}} = \lim_{t\to\infty}\frac{1}{5e^{5x}}|_1^t = \frac{1}{5e^5}$. Thus by the comparison theorem, since the larger integral converges so does the smaller. Hence $\int_1^\infty \frac{dx}{x+e^{5x}}$ converges by comparison to $\int_1^\infty \frac{dx}{e^{5x}}$.

11. $F = \int \rho g dA$. We orient the semicircular plate so the origin passes through the center of the diameter. Thus the depth d of an arbitrary strip of oil is $d = y$ and the area is $A = 2xdy$ where $x^2 + y^2 = 1$. Hence $A = 2\sqrt{1-y^2}dy$. This yields $F = \int_0^1 2\rho g y\sqrt{1-y^2}\,dy$. Integrate with u substitution with $u = 1 - y^2$ we obtain $F = \frac{2}{3}\rho g = 6000N$.

12. Let $y = y(t)$ be the amount of salt in the tank at time t. Since we are starting with pure water, $y(0) = 0$. Now,
$\frac{dy}{dt} = \left(0.01\frac{kg}{L}\right)\left(20\frac{L}{m}\right) - \left(\frac{y}{250}\frac{kg}{L}\right)\left(20\frac{L}{m}\right)$.
Thus $\frac{dy}{dt} = 0.2\frac{kg}{m} - \frac{2}{25}y\frac{kg}{m}$. We can solve this equation as linear or separable. I will solve it as separable: $\frac{dy}{dt} = \frac{5-2y}{25}$, thus $\frac{dy}{5-2y} = \frac{1}{25}dt$. Integrating both sides yields
$-\frac{1}{2}\ln(5-2y) = \frac{1}{25}t + C$. Since $y(0) = 0$, this gives $C = -\frac{1}{2}\ln(5)$. $-\frac{1}{2}\ln(5-2y) = \frac{1}{25}t - \frac{1}{2}\ln(5)$. Thus $\ln(5-2y) = -\frac{2}{25}t + \ln(5)$. Solve for y:
$5 - 2y = e^{-2t/25+\ln 5} = 5e^{-2t/25}$.
Hence $y = \dfrac{5 - 5e^{-2t/25}}{2}$ kg of salt.

13. $\frac{x+2}{x^2(x^2+1)}$ has a partial fraction decomposition of the form $\frac{x+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$. Thus
$x + 2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$.
$x + 2 = (A+C)x^3 + (B+D)x^2 + Ax + B$. Equate coefficients: $A + C = 0$, $B + D = 0$, $A = 1$ and $B = 2$, which gives $C = -1$ and $D = -2$. Thus
$\int \frac{x+2}{x^2(x^2+1)}\,dx = \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{-x-2}{x^2+1}\right)dx$
$= \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{-x}{x^2+1} - \frac{2}{x^2+1}\right)dx$
$= \ln|x| - \frac{2}{x} - \frac{1}{2}\ln(x^2+1) - 2\arctan x + C$.

14. $x\frac{dy}{dx} = x(\ln x)^2 + y$ is a linear differential equation. Getting this in linear form gives
$\frac{dy}{dx} - \frac{1}{x}y = (\ln x)^2$. Now the integrating factor is $I(x) = e^{\int -1/x\,dx} = e^{-\ln x} = \frac{1}{x}$.
Hence $\frac{1}{x}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = \frac{1}{x}(\ln x)^2$.
Thus $\frac{d}{dx}(y\frac{1}{x}) = \frac{1}{x}(\ln x)^2$. Integrate both sides:
$\int \frac{d}{dx}(y\frac{1}{x})\,dx = \int \frac{1}{x}(\ln x)^2\,dx$. Using a u-substitution with $u = \ln x$, we obtain
$y\frac{1}{x} = \frac{(\ln x)^3}{3} + C$, thus $y = x\left(\frac{(\ln x)^3}{3} + C\right)$.

15. The curves $y = \sqrt{x}$ and $y = x^3$ intersect at $x = 0$ and $x = 1$. Hence
$A = \int_0^1 (\sqrt{x} - x^3)\,dx = \frac{5}{12}$.
$\overline{x} = \frac{1}{A}\int_0^1 x(\sqrt{x} - x^3)\,dx$
$= \frac{12}{5}\int_0^1 (x^{3/2} - x^4)\,dx = \frac{12}{25}$.
$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left((\sqrt{x})^2 - (x^3)^2\right)dx$
$= \frac{12}{10}\int_0^1 (x - x^6)\,dx = \frac{3}{7}$.
Thus the centroid is $\left(\frac{12}{25}, \frac{3}{7}\right)$