Math 152 Spring 2009 Exam III Solutions-Form A

1. d: $T_3(x) = \sum\limits_{i=0}^{3} \dfrac{f^{(i)}(1)}{i!}(x-1)^i$

   $= f(1)+f'(1)(x-1)+\dfrac{f''(1)}{2!}(x-1)^2+\dfrac{f'''(1)}{3!}(x-1)^3$
   $= x-1-\dfrac{1}{2}(x-1)^2+\dfrac{1}{3}(x-1)^3$. Substitute $x=2$ into $T_3(x)$ yields $T_3(2) = \dfrac{5}{6}$.

2. a: Taylor's Inequality says an upper bound on the absolute value of the remainder in using $T_n(x)$ to approximate $f(x)$ in an interval containing a is $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$ where

   $M = max|f^{(n+1)}(x)|$ for x in an interval containing a. Here $n=3$ and $a = \dfrac{\pi}{3}$.

   Thus $|R_3(x)| \leq \dfrac{M}{4!}\left|x-\dfrac{\pi}{3}\right|^4$,

   where $M = max|f^{(4)}(x)|$ for $0 \leq x \leq \dfrac{2\pi}{3}$.

   $f^{(4)}(x) = \sin x$, hence the maximum of $|f^{(4)}(x)|$ occures when $x = \dfrac{\pi}{2}$, yielding $M=1$. Thus

   $|R_3(x)| \leq \dfrac{1}{4!}\left|x-\dfrac{\pi}{3}\right|^4$. Now if $0 \leq x \leq \dfrac{2\pi}{3}$, the maximum of $|x-\dfrac{\pi}{3}|$ is $\dfrac{\pi}{3}$. Thus

   $|R_3(x)| \leq \dfrac{1}{4!}\left(\dfrac{\pi}{3}\right)^4$.

3. c: Recall $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n$, where s_n is the sequence of partial sums. We are given that

   $s_n = 4+\ln(2n)-\ln(n+1)$, thus

   $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty}(4+\ln(2n)-\ln(n+1))$

   $= \lim\limits_{n\to\infty}\left(4+\ln\dfrac{2n}{n+1}\right) = 4+\ln 2$

4. c: The Ratio Test is inconclusive for $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\sqrt{\ln n}}$ since

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{\sqrt{\ln(n+1)}}\frac{\sqrt{\ln n}}{(-1)^n}\right| = 1.$$

5. a: $\sum\limits_{n=1}^{\infty}\dfrac{2+\cos n}{n^2} \leq \sum\limits_{n=1}^{\infty}\dfrac{3}{n^2}$. Since $\sum\limits_{n=1}^{\infty}\dfrac{3}{n^2}$ is a convergent p-series, $\sum\limits_{n=1}^{\infty}\dfrac{2+\cos n}{n^2}$ also converges by the Comparison Test.

6. d: $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n+3^n}{5^n} = \sum\limits_{n=0}^{\infty}\left(\dfrac{(-1)^n}{5^n}+\dfrac{3^n}{5^n}\right) = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{5^n}+\sum\limits_{n=0}^{\infty}\dfrac{3^n}{5^n}$

   $= \sum\limits_{n=0}^{\infty}\left(-\dfrac{1}{5}\right)^n + \sum\limits_{n=0}^{\infty}\left(\dfrac{3}{5}\right)^n$

   $= \dfrac{1}{1+1/5}+\dfrac{1}{1-3/5} = \dfrac{10}{3}$.

7. c: $S_n = \sum\limits_{i=1}^{n}\left(\sin\dfrac{1}{i}-\sin\dfrac{1}{i+2}\right)$

   $= \sin 1 - \sin\dfrac{1}{3}+\sin\dfrac{1}{2}-\sin\dfrac{1}{4}+\sin\dfrac{1}{3}-\sin\dfrac{1}{5}+...+\sin\dfrac{1}{n-1}-\sin\dfrac{1}{n+1}+\sin\dfrac{1}{n}-\sin\dfrac{1}{n+2}$

   $= \sin(1)+\sin\dfrac{1}{2}-\sin\dfrac{1}{n+1}-\sin\dfrac{1}{n+2}$. Now, $\sum\limits_{n=1}^{\infty}\left(\sin\dfrac{1}{n}-\sin\dfrac{1}{n+2}\right) = \lim\limits_{n\to\infty} s_n$

   $= \lim\limits_{n\to\infty}\left(\sin(1)+\sin\dfrac{1}{2}-\sin\dfrac{1}{n+1}-\sin\dfrac{1}{n+2}\right)$

   $= \sin(1)+\sin\dfrac{1}{2}$.

8. c: Collect all variables on the left hand side of the equation and complete the squares:

   $x^2-2x+y^2+4y+z^2-6z=0$. Complete the three squares:

   $x^2-2x+1+y^2+4y+4+z^2-6z+9 = 14 \Rightarrow (x-1)^2+(y+2)^2+(z-3)^2 = 14$. Thus the center of the sphere is $(1,-2,3)$ and the radius is $\sqrt{14}$

9. d. Using the known Maclaurin Series for

   $\sin x = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$,

   $f(x) = x^2\sin x = x^2\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

   $= \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+3}}{(2n+1)!}$

10. a: The series $\sum_{n=1}^{\infty} \frac{n}{4n+100}$ diverges by the test for divergence because $\lim_{n\to\infty} \frac{n}{4n+100} = \frac{1}{4} \neq 0$. All of the other statements are false.

11. b: Apply the Ratio Test to the series $\sum_{n=1}^{\infty} \frac{5^n(x-3)^n}{n!}$:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{5^{n+1}(x-3)^{n+1}}{(n+1)!} \frac{n!}{5^n(x-3)^n}\right|$$

$= \lim_{n\to\infty} \left|\frac{5(x-3)}{(n+1)}\right| = 0 < 1$ for all x. Hence the interval of convergence is all real numbers, yielding a radius of convergence of $R = \infty$.

12. Use the Ratio Test: for $\sum_{n=2}^{\infty} \frac{(-3)^n(x-1)^n}{\sqrt{n}}$:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(-3)^{n+1}(x-1)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(-3)^n(x-1)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{-3(x-1)}{\sqrt{n+1}} \frac{\sqrt{n}}{1}\right|$$

$= |-3(x-1)| = |-3||(x-1)| = 3|x-1|$. Now the Ratio Test says if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges. Hence

$3|x-1| < 1$, so $|x-1| < \frac{1}{3}$, hence $R = \frac{1}{3}$. Now test the endpoints of the interval:

$|x-1| < \frac{1}{3}$ yields $\frac{2}{3} < x < \frac{4}{3}$. Testing $x = \frac{2}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n(-1/3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges by p-series.

Testing $x = \frac{4}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n(1/3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $\frac{2}{3} < x \leq \frac{4}{3}$.

13. $f(x) = \ln(2 - x^3)$, thus $f'(x) = \frac{-3x^2}{2-x^3} = \frac{-3x^2}{2(1-x^3/2)} = \frac{-3x^2}{2} \frac{1}{1-\frac{x^3}{2}}$

$= \frac{-3x^2}{2} \sum_{n=0}^{\infty} \left(\frac{x^3}{2}\right)^n$, where $\left|\frac{x^3}{2}\right| < 1$,

hence $R = \sqrt[3]{2}$.

Now, $\frac{-3x^2}{2} \sum_{n=0}^{\infty} \left(\frac{x^3}{2}\right)^n = \sum_{n=0}^{\infty} -3\frac{x^{3n+2}}{2^{n+1}}$.

Thus $\ln(2-x^3) = \int \sum_{n=0}^{\infty} -3\frac{x^{3n+2}}{2^{n+1}}\, dx$

$= C + \sum_{n=0}^{\infty} -3\frac{x^{3n+3}}{2^{n+1}(3n+3)}$.

$= C - \sum_{n=0}^{\infty} \frac{x^{3n+3}}{2^{n+1}(n+1)}$

Find C by choosing $x = 0$, yielding $C = \ln 2$.

Thus $\ln(2-x^3) = \ln 2 - \sum_{n=0}^{\infty} \frac{x^{3n+3}}{2^{n+1}(n+1)}$.

14. (a) First, I will apply the Alternating Series Test to determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges with

$a_n = \frac{1}{n \ln n}$:

(i) $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$, hence $a_{n+1} < a_n$.

(ii) $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$.

Therefore both conditions of the AST are met, so the series converges. To test absolute convergence, we look at $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n \ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$. Use the integral test: $\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} (\ln(\ln t) - \ln(\ln 2)) = \infty$ (u-sub with $u = \ln x$). Thus $\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges, hence $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges, but not absolutely.

(b) Since $\sum_{n=2}^{\infty} \frac{1}{n \ln n + \sqrt{n}}$ is a positive series, and since this is part (b) of a problem, that is a hint that I will somehow use part (a). In part (a), I showed $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverged. Thus I will use the Limit Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1/(n\ln n + \sqrt{n})}{1/(n \ln n)}$

$= \lim\limits_{n\to\infty} \dfrac{n\ln n}{n \ln n + \sqrt{n}}$. Divide the numerator and denominator by $n \ln n$.

$= \lim\limits_{n\to\infty} \dfrac{1}{1 + \dfrac{\sqrt{n}}{n\ln n}} = \lim\limits_{n\to\infty} \dfrac{1}{1 + \dfrac{1}{\sqrt{n}\ln n}} = 1$. Thus since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n}$ diverges, so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n + \sqrt{n}}$.

15. The Taylor Series for $f(x) = \dfrac{1}{x}$ at $x = 3$ is

$f(x) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(3)}{n!}(x-3)^n$. We need to find a formula for $f^{(n)}(3)$.

$f(x) = \dfrac{1}{x}$

$f'(x) = -\dfrac{1}{x^2}$

$f''(x) = \dfrac{2}{x^3}$

$f'''(x) = -\dfrac{3 \cdot 2}{x^4}$

$f^{(4)}(x) = \dfrac{4 \cdot 3 \cdot 2}{x^5}$

$f^{(5)}(x) = -\dfrac{5 \cdot 4 \cdot 3 \cdot 2}{x^6}$

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$f^{(n)}(x) = \dfrac{(-1)^n n!}{x^{n+1}}$

Substitute $x = 3$: $f^{(n)}(3) = \dfrac{(-1)^n n!}{3^{n+1}}$. Substitute this in above, we obtain

$f(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n n!}{3^{n+1} n!}(x-3)^n$

$= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{3^{n+1}}(x-3)^n$

16. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{3^n n!}$:

a.) We will use the Ratio Test to establish convergence:

$\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$

$= \lim\limits_{n\to\infty} \left| \dfrac{(-1)^{n+1}}{3^{n+1}(n+1)!} \dfrac{3^n n!}{(-1)^n} \right|$

$= \lim\limits_{n\to\infty} \left| \dfrac{-1}{3(n+1)} \right| = 0 < 1$, thus the series converges absolutely by the Ratio Test.

b.) $s_4 = \sum\limits_{i=1}^{4} \dfrac{(-1)^i}{3^i i!}$

$= -\dfrac{1}{3} + \dfrac{1}{3^2 2!} - \dfrac{1}{3^3 3!} + \dfrac{1}{3^4 4!}$

c.) Using the Remainder Estimate formula for Alternating Series:

$|R_n| < a_{n+1}$ where a_{n+1} is the positive part of the $(n+1)^{th}$ term. Since $n = 4$:

$|R_n| < a_5 = \dfrac{1}{3^5 5!}$.