Math 152 Spring 2009 Exam III Solutions-Form B

1. d: Recall $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n$, where s_n is the sequence of partial sums. We are given that

   $s_n = 4 + \ln(n+1) - \ln(2n)$, thus

   $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} (4 + \ln(n+1) - \ln(2n))$

   $= \lim\limits_{n\to\infty} \left(4 + \ln\frac{n+1}{2n}\right) = 4 + \ln\frac{1}{2}$

2. e: Collect all variables on the left hand side of the equation and complete the squares:

   $x^2 + 2x + y^2 - 4y + z^2 - 6z = 0$. Complete the three squares:

   $x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 = 14 \Rightarrow (x+1)^2 + (y-2)^2 + (z-3)^2 = 14$. Thus the center of the sphere is $(-1, 2, 3)$ and the radius is $\sqrt{14}$

3. a: $T_3(x) = \sum\limits_{i=0}^{3} \frac{f^{(i)}(1)}{i!}(x-1)^i$

   $= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$
   $= x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$. Substitute $x = 3$ into $T_3(x)$ yields $T_3(3) = \frac{8}{3}$.

4. a: $\sum\limits_{n=0}^{\infty} \frac{(-1)^n + 2^n}{5^n} = \sum\limits_{n=0}^{\infty} \left(\frac{(-1)^n}{5^n} + \frac{2^n}{5^n}\right) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{5^n} + \sum\limits_{n=0}^{\infty} \frac{2^n}{5^n}$

   $= \sum\limits_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n + \sum\limits_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

   $= \frac{1}{1 + 1/5} + \frac{1}{1 - 2/5} = \frac{5}{2}$.

5. c: The series $\sum\limits_{n=0}^{\infty} 2\left(\frac{3}{5}\right)^n$ is a convergent geometric series with $r = \frac{3}{5}$. All of the other statements are false.

6. c: Taylor's Inequality says an upper bound on the absolute value of the remainder in using $T_n(x)$ to approximate $f(x)$ in an interval containing a is $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ where

   $M = max|f^{(n+1)}(x)|$ for x in an interval containing a. Here $n = 3$ and $a = \frac{\pi}{3}$.

   Thus $|R_3(x)| \le \frac{M}{4!}\left|x - \frac{\pi}{3}\right|^4$,

   where $M = max|f^{(4)}(x)|$ for $0 \le x \le \frac{2\pi}{3}$.

   $f^{(4)}(x) = \sin x$, hence the maximum of $|f^{(4)}(x)|$ occures when $x = \frac{\pi}{2}$, yielding $M = 1$. Thus

   $|R_3(x)| \le \frac{1}{4!}\left|x - \frac{\pi}{3}\right|^4$. Now if $0 \le x \le \frac{2\pi}{3}$, the maximum of $|x - \frac{\pi}{3}|$ is $\frac{\pi}{3}$. Thus

   $|R_3(x)| \le \frac{1}{4!}\left(\frac{\pi}{3}\right)^4$.

7. a: Apply the Ratio Test to the series $\sum\limits_{n=1}^{\infty} \frac{n!(x-3)^n}{5^n}$:

   $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(n+1)!(x-3)^{n+1}}{5^{n+1}} \frac{5^n}{n!(x-3)^n}\right|$

   $= \lim\limits_{n\to\infty} \left|\frac{(x-3)(n+1)}{5}\right| = \infty > 1$ for all x. Hence the interval of convergence is $\{3\}$, yielding a radius of convergence of $R = 0$.

8. e: The Ratio Test is inconclusive for $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}$ since

   $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(-1)^{n+1}}{\sqrt{\ln(n+1)}} \frac{\sqrt{\ln n}}{(-1)^n}\right| = 1$.

9. e: $S_n = \sum\limits_{i=1}^{n} \left(\sin\frac{1}{i} - \sin\frac{1}{i+2}\right)$

   $= \sin 1 - \sin\frac{1}{3} + \sin\frac{1}{2} - \sin\frac{1}{4} + \sin\frac{1}{3} - \sin\frac{1}{5} + ... + \sin\frac{1}{n-1} - \sin\frac{1}{n+1} + \sin\frac{1}{n} - \sin\frac{1}{n+2}$

   $= \sin(1) + \sin\frac{1}{2} - \sin\frac{1}{n+1} - \sin\frac{1}{n+2}$. Now, $\sum\limits_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+2}\right) = \lim\limits_{n\to\infty} s_n$

   $= \lim\limits_{n\to\infty} \left(\sin(1) + \sin\frac{1}{2} - \sin\frac{1}{n+1} - \sin\frac{1}{n+2}\right)$

   $= \sin(1) + \sin\frac{1}{2}$.

10. b: $\sum\limits_{n=1}^{\infty} \dfrac{4+\cos n}{n^2} \leq \sum\limits_{n=1}^{\infty} \dfrac{5}{n^2}$. Since $\sum\limits_{n=1}^{\infty} \dfrac{5}{n^2}$ is a convergent p-series, $\sum\limits_{n=1}^{\infty} \dfrac{4+\cos n}{n^2}$ also converges by the Comparison Test.

11. e. Using the known Maclaurin Series for

$\cos x = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$,

$f(x) = x^2 \cos x = x^2 \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$

$= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{(2n)!}$

12. Use the Ratio Test: for $\sum\limits_{n=2}^{\infty} \dfrac{(-4)^n (x-2)^n}{\sqrt{n}}$:

$\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$

$= \lim\limits_{n\to\infty} \left| \dfrac{(-4)^{n+1}(x-2)^{n+1}}{\sqrt{n+1}} \dfrac{\sqrt{n}}{(-4)^n (x-2)^n} \right|$

$= \lim\limits_{n\to\infty} \left| \dfrac{-4(x-2)}{\sqrt{n+1}} \dfrac{\sqrt{n}}{1} \right|$

$= |-4(x-2)| = |-4||(x-2)| = 4|x-2|$. Now the Ratio Test says if $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ converges. Hence

$4|x-2| < 1$, so $|x-2| < \dfrac{1}{4}$, hence $R = \dfrac{1}{4}$. Now test the endpoints of the interval:

$|x-2| < \dfrac{1}{4}$ yields $\dfrac{7}{4} < x < \dfrac{9}{4}$. Testing $x = \dfrac{7}{4}$: $\sum\limits_{n=1}^{\infty} \dfrac{(-4)^n(-1/4)^n}{\sqrt{n}} = \sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ which diverges by p-series.

Testing $x = \dfrac{9}{4}$: $\sum\limits_{n=1}^{\infty} \dfrac{(-4)^n(1/4)^n}{\sqrt{n}} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $\dfrac{7}{4} < x \leq \dfrac{9}{4}$.

13. $f(x) = \ln(5 - x^3)$, thus $f'(x) = \dfrac{-3x^2}{5 - x^3} = \dfrac{-3x^2}{5(1 - x^3/5)} = \dfrac{-3x^2}{5} \dfrac{1}{1 - \dfrac{x^3}{5}}$

$= \dfrac{-3x^2}{5} \sum\limits_{n=0}^{\infty} \left(\dfrac{x^3}{5} \right)^n$, where $\left| \dfrac{x^3}{5} \right| < 1$,

hence $R = \sqrt[3]{5}$.

Now, $\dfrac{-3x^2}{5} \sum\limits_{n=0}^{\infty} \left(\dfrac{x^3}{5} \right)^n = \sum\limits_{n=0}^{\infty} -3 \dfrac{x^{3n+2}}{5^{n+1}}$.

Thus $\ln(5 - x^3) = \displaystyle\int \sum_{n=0}^{\infty} -3 \frac{x^{3n+2}}{5^{n+1}}\, dx$

$= C + \sum\limits_{n=0}^{\infty} -3 \dfrac{x^{3n+3}}{5^{n+1}(3n+3)}$

$= C - \sum\limits_{n=0}^{\infty} \dfrac{x^{3n+3}}{5^{n+1}(n+1)}$

Find C by choosing $x = 0$, yielding $C = \ln 5$.

Thus $\ln(5 - x^3) = \ln 5 - \sum\limits_{n=0}^{\infty} \dfrac{x^{3n+3}}{5^{n+1}(n+1)}$.

14. (a) First, I will apply the Alternating Series Test to determine whether $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n \ln n}$ converges with

$a_n = \dfrac{1}{n \ln n}$:

(i) $\dfrac{1}{(n+1)\ln(n+1)} < \dfrac{1}{n \ln n}$, hence $a_{n+1} < a_n$.

(ii) $\lim\limits_{n\to\infty} \dfrac{1}{n \ln n} = 0$.

Therefore both conditions of the AST are met, so the series converges. To test absolute convergence, we look at $\sum\limits_{n=2}^{\infty} \left| \dfrac{(-1)^n}{n \ln n} \right| = \sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$. Use the integral test: $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} (\ln(\ln t) - \ln(\ln 2)) = \infty$ (u-sub with $u = \ln x$). Thus $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges, hence $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n \ln n}$ converges, but not absolutely.

(b) Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n + \sqrt{n}}$ is a positive series, and since this is part (b) of a problem, that is a hint that I will somehow use part (a). In part (a), I showed $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverged. Thus I will use the Limit Comparison Test with $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$.

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1/(n \ln n + \sqrt{n})}{1/(n \ln n)}$

$= \lim\limits_{n\to\infty} \dfrac{n \ln n}{n \ln n + \sqrt{n}}$. Divide the numerator and denominator by $n \ln n$.

$= \lim\limits_{n\to\infty} \dfrac{1}{1 + \dfrac{\sqrt{n}}{n \ln n}} = \lim\limits_{n\to\infty} \dfrac{1}{1 + \dfrac{1}{\sqrt{n} \ln n}} = 1$. Thus since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges, so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n + \sqrt{n}}$.

15. The Taylor Series for $f(x) = \dfrac{1}{x}$ at $x = 4$ is

$f(x) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(4)}{n!}(x-4)^n$. We need to find a formula for $f^n(4)$.

$f(x) = \dfrac{1}{x}$

$f'(x) = -\dfrac{1}{x^2}$

$f''(x) = \dfrac{2}{x^3}$

$f'''(x) = -\dfrac{3 \cdot 2}{x^4}$

$f^{(4)}(x) = \dfrac{4 \cdot 3 \cdot 2}{x^5}$

$f^{(5)}(x) = -\dfrac{5 \cdot 4 \cdot 3 \cdot 2}{x^6}$

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$f^{(n)}(x) = \dfrac{(-1)^n n!}{x^{n+1}}$

Substitute $x = 4$: $f^{(n)}(4) = \dfrac{(-1)^n n!}{4^{n+1}}$. Substitute this in above, we obtain

$f(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n n!}{4^{n+1} n!}(x-4)^n$

$= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{4^{n+1}}(x-4)^n$

16. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{2^n n!}$:

a.) We will use the Ratio Test to establish convergence:

$\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$

$= \lim\limits_{n\to\infty} \left| \dfrac{(-1)^{n+1}}{2^{n+1}(n+1)!} \dfrac{2^n n!}{(-1)^n} \right|$

$= \lim\limits_{n\to\infty} \left| \dfrac{-1}{2(n+1)} \right| = 0 < 1$, thus the series converges absolutely by the Ratio Test.

b.) $s_4 = \sum\limits_{i=1}^{4} \dfrac{(-1)^i}{2^i i!}$

$= -\dfrac{1}{2} + \dfrac{1}{2^2 2!} - \dfrac{1}{2^3 3!} + \dfrac{1}{2^4 4!}$

c.) Using the Remainder Estimate formula for Alternating Series:

$|R_n| < a_{n+1}$ where a_{n+1} is the positive part of the $(n+1)^{th}$ term. Since $n = 4$:

$|R_n| < a_5 = \dfrac{1}{2^5 5!}$.