Fall 2009 Math 152

Exam II Version A Solutions

1. **E** Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$. Substituting into the integral yields $\int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta} 2\sec\theta\tan\theta\,d\theta = \int 2\sqrt{\sec^2\theta - 1}\,\tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta$.

2. **C** There is a repeated linear and repeated quadratic factor in the denominator; therefore the correct form is $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+5} + \frac{Fx+G}{(x^2+5)^2}$.

3. **C** The integral is defined as $\lim_{a\to\infty}\int_0^a xe^{-2x}\,dx$. Integrate by parts, with $u = x$, $dv = e^{-2x}\,dx$. Then $du = dx$, $v = -\frac{1}{2}e^{-2x}$. The integral is $= \lim_{a\to\infty} -\frac{1}{2}xe^{-2x}\Big|_0^a - \int_0^a -\frac{1}{2}e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^a = \lim_{a\to\infty} -\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} - 0 + \frac{1}{4}$. The first term approaches 0 by L'Hospital's Rule; the second term approaches 0 from the graph of the exponential function. Therefore, the limit is $\frac{1}{4}$.

4. **D** Integral (I) converges by Comparison with $\int_0^\infty e^{-x}\,dx$. Integral (II) converges by Comparison with $\int_1^\infty \frac{1}{x^2}\,dx$. Integral (III) diverges by direct integration $(\lim_{t\to\infty} \ln|1+t| - \ln 2)$.

5. **A** If $p \neq 1$, $\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-p}\,dx$
$= \lim_{a\to 0^+} \frac{1}{-p+1}x^{-p+1}\Big|_a^1$
$= \lim_{a\to 0^+} \frac{1}{1-p}(1 - a^{1-p})$, which is finite when $0 < p < 1$. If $p = 1$, $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \ln x\big|_a^1 = \lim_{a\to 0^+}(-\ln a) = +\infty$. Therefore, the integral converges only when $0 < p < 1$.

6. **C** $\frac{dy}{dx} = \frac{1}{2}(x^2+2)^{1/2}(2x) = x\sqrt{x^2+2}$. Therefore, $s = \int_0^1 \sqrt{1 + x^2(x^2+2)}\,dx =$
$= \int_0^1 \sqrt{x^4 + 2x^2 + 1}\,dx = \int_0^1 (x^2+1)\,dx$
$= \frac{1}{3}x^3 + x\Big|_0^1 = \frac{4}{3}$.

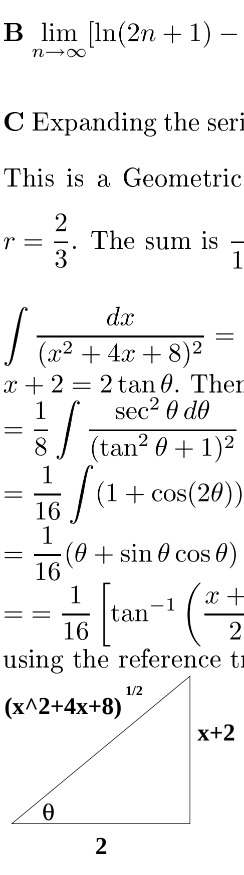
7. **A** $S = \int_a^b 2\pi x\,ds$, where $x = 2\cos t$, $ds = \sqrt{(-2\sin t)^2 + (3\cos t)^2}$, and $[a,b] = \left[0, \frac{\pi}{2}\right]$. Therefore, the surface area is $\int_0^{\pi/2} 2\pi(2\cos t)\sqrt{4\sin^2 t + 9\cos^2 t}\,dt$
$= 4\pi\int_0^{\pi/2} \cos t\sqrt{4\sin^2 t + 9(1 - \sin^2 t)} =$
$4\pi\int_0^{\pi/2} \cos t\sqrt{9 - 5\sin^2 t}\,dt$.

8. **D** Using properties of limits, we obtain $\frac{(2)(-3)}{2 + 3(-3)^2} = -\frac{6}{29}$.

9. **B** $\lim_{n\to\infty}[\ln(2n+1) - \ln(n)] = \lim_{n\to\infty} \ln\left(\frac{2n-1}{n}\right) = \ln 2$

10. **C** Expanding the series yields $\frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$. This is a Geometric series with $a = \frac{4}{3}$ and $r = \frac{2}{3}$. The sum is $\frac{\frac{4}{3}}{1 - \frac{2}{3}} = 4$.

11. $\int \frac{dx}{(x^2+4x+8)^2} = \int \frac{dx}{[(x+2)^2+4]^2}$. Let $x + 2 = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$.
$= \frac{1}{8}\int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta + 1)^2} = \frac{1}{8}\int \cos^2\theta\,d\theta$
$= \frac{1}{16}\int (1 + \cos(2\theta))d\theta = \frac{1}{16}(\theta + \frac{1}{2}\sin(2\theta) + C$
$= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C$
$= = \frac{1}{16}\left[\tan^{-1}\left(\frac{x+2}{2}\right) + \frac{2(x+2)}{x^2+4x+8}\right] + C$
using the reference triangle below.



12. Using partial fractions, $\int \frac{2x^2+2x+1}{x^2(x^2+1)}\,dx = \int \left(\frac{2}{x}+\frac{1}{x^2}+\frac{-2x+1}{x^2+1}\right)dx$ $= 2\ln|x| - \frac{1}{x} - \ln(x^2+1) + \tan^{-1}(x) + C.$

13. $\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$, so the surface area is given by $S = \int_0^a 2\pi y\,ds = 2\pi\int_0^a \sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx$ $= 2\pi\int_0^a dx = 2\pi a = \frac{3}{2}$, so $a = \frac{3}{4\pi}$.

14. .

(a) $\lim_{n\to\infty} n^2e^{-2n} = \lim_{n\to\infty}\frac{n^2}{e^{2n}}$. Apply L'Hospital's Rule to compute the limit of the real-valued function $f(x) = \frac{x^2}{e^{2x}}$. $\lim_{x\to\infty}\frac{x^2}{e^{2x}} = \lim_{x\to\infty}\frac{2x}{2e^{2x}} = \lim_{x\to\infty}\frac{2}{4e^{2x}} = 0$. Therefore, the sequence also approaches 0.

(b) Let $a_n = \frac{n}{2n+3}$. $\lim_{n\to\infty} a_n = \frac{1}{2}(\neq 0)$, therefore, by the Test for Divergence, the series $\sum_{n=1}^{\infty}\frac{n}{2n+3}$ is divergent.

15. .

(a) $0 \le \frac{\sin^2(\sqrt{n})}{n(n+3)} \le \frac{1}{n^2+3n} \le \frac{1}{n^2}$. The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$converges since it is a p-series with $p=2$. Therefore, by the Comparison Test, $\sum_{n=1}^{\infty}\frac{\sin^2(\sqrt{n})}{n(n+3)}$ is convergent.

(b) The terms of the series are all positive. Let $a_n = \frac{1}{1+n^{1/3}}$ and $b_n = \frac{1}{n^{1/3}}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{1/3}}{1+n^{1/3}} = 1$, which means the series $\sum a_n$and $\sum b_n$ are either both convergent or both divergent. But $\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ is divergent since it is a p-series with $p = \frac{1}{3}$, therefore $\sum_{n=1}^{\infty}\frac{1}{1+n^{1/3}}$ is divergent by the Limit Comparison Test.

16. Since the length of C is L, $\int_0^1 \sqrt{(f'(t))^2+(g'(t))^2}\,dt = L$. For the new curve, $\frac{dx}{dt} = g'(1-t)(-1)$ and $\frac{dy}{dt} = f'(1-t)(-1)$. Therefore, the length of the new curve is given by $s = \int_0^1 \sqrt{(g'(1-t))^2+(f'(1-t))^2}\,dt$. Let $u = 1-t$. Then $du = -dt$, or $dt = -du$. When $t=0$, $u=1$, and when $t=1$, $u=0$. Therefore,$s = -\int_1^0 \sqrt{(g'(u))^2+(f'(u))^2}\,du$ $= \int_0^1 \sqrt{(f'(u))^2+(g'(u))^2}\,du = L.$