

Fall 2009 Math 152

Exam II Version B Solutions

1. **B** Let $x = 3 \sec \theta$. Then $dx = 3 \sec \theta \tan \theta d\theta$. Substituting into the integral yields $\int \frac{\sqrt{9 \sec^2 \theta - 9}}{3 \sec \theta} 3 \sec \theta \tan \theta d\theta = \int 3 \sqrt{\sec^2 \theta - 1} \tan \theta d\theta = 3 \int \tan^2 \theta d\theta$.

2. **E** There is a repeated linear and repeated quadratic factor in the denominator; therefore the correct form is $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+5} + \frac{Fx+G}{(x^2+5)^2}$.

3. **A** The integral is defined as $\lim_{a \rightarrow \infty} \int_0^a x e^{-3x} dx$. Integrate by parts, with $u = x$, $dv = e^{-3x} dx$. Then $du = dx$, $v = -\frac{1}{3} e^{-3x}$. The integral is $= \lim_{a \rightarrow \infty} -\frac{1}{3} x e^{-3x} \Big|_0^a - \int_0^a -\frac{1}{3} e^{-3x} dx = -\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} \Big|_0^a = \lim_{a \rightarrow \infty} -\frac{1}{3} a e^{-3a} - \frac{1}{9} e^{-3a} - 0 + \frac{1}{9}$. The first term approaches 0 by L'Hospital's Rule; the second term approaches 0 from the graph of the exponential function. Therefore, the limit is $\frac{1}{9}$.

4. **A** Integral (I) diverges by direct integration ($\lim_{t \rightarrow \infty} \ln |1+t| - \ln 2$). Integral (II) converges by Comparison with $\int_1^\infty \frac{1}{x^2} dx$. Integral (III) converges by Comparison with $\int_0^\infty e^{-x} dx$.

5. **D** If $p \neq 1$, $\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-p} dx = \lim_{a \rightarrow 0^+} \frac{1}{-p+1} x^{-p+1} \Big|_a^1 = \lim_{a \rightarrow 0^+} \frac{1}{1-p} (1 - a^{1-p})$, which is finite when $0 < p < 1$. If $p = 1$, $\int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \ln x \Big|_a^1 = \lim_{a \rightarrow 0^+} (-\ln a) = +\infty$. Therefore, the integral converges only when $0 < p < 1$.

6. **A** $\frac{dy}{dx} = \frac{1}{2}(x^2 + 2)^{1/2}(2x) = x\sqrt{x^2 + 2}$.

Therefore, $s = \int_0^2 \sqrt{1 + x^2(x^2 + 2)} dx = \int_0^2 \sqrt{x^4 + 2x^2 + 1} dx = \int_0^2 (x^2 + 1) dx = \frac{1}{3}x^3 + x \Big|_0^2 = \frac{14}{3}$.

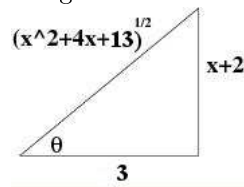
7. **E** $S = \int_a^b 2\pi x ds$, where $x = 3 \cos t$, $ds = \sqrt{(-3 \sin t)^2 + (2 \cos t)^2}$, and $[a, b] = [0, \frac{\pi}{2}]$. Therefore, the surface area is $\int_0^{\pi/2} 2\pi(3 \cos t) \sqrt{9 \sin^2 t + 4 \cos^2 t} dt = 6\pi \int_0^{\pi/2} \cos t \sqrt{9 \sin^2 t + 4(1 - \sin^2 t)} dt = 6\pi \int_0^{\pi/2} \cos t \sqrt{4 + 5 \sin^2 t} dt$.

8. **B** Using properties of limits, we obtain $\frac{(3)(-2)}{2 + 3(-2)^2} = -\frac{6}{14} = -\frac{3}{7}$.

9. **A** $\lim_{n \rightarrow \infty} [\ln(n) - \ln(2n + 1)] = \lim_{n \rightarrow \infty} \ln \left(\frac{n}{2n + 1} \right) = \ln \left(\frac{1}{2} \right)$.

10. **E** Expanding the series yields $\frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots$. This is a Geometric series with $a = \frac{2}{9}$ and $r = \frac{2}{3}$. The sum is $\frac{\frac{2}{9}}{1 - \frac{2}{3}} = \frac{2}{3}$.

11. $\int \frac{dx}{(x^2 + 4x + 13)^2} = \int \frac{dx}{[(x+2)^2 + 9]^2}$. Let $x+2 = 3 \tan \theta$. Then $dx = 3 \sec^2 \theta d\theta$. $= \frac{1}{27} \int \frac{\sec^2 \theta d\theta}{(\tan^2 \theta + 1)^2} = \frac{1}{27} \int \cos^2 \theta d\theta = \frac{1}{54} \int (1 + \cos(2\theta)) d\theta = \frac{1}{54} (\theta + \frac{1}{2} \sin(2\theta)) + C = \frac{1}{54} (\theta + \sin \theta \cos \theta) + C = \frac{1}{54} \left[\tan^{-1} \left(\frac{x+2}{3} \right) + \frac{3(x+2)}{x^2 + 4x + 13} \right] + C$ using the reference triangle below.



12. Using partial fractions,

$$\int \frac{x^2 + x + 2}{x^2(x^2 + 1)} dx = \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{-x - 1}{x^2 + 1} \right) dx$$

$$= \ln|x| - \frac{2}{x} - \frac{1}{2} \ln(x^2 + 1) - \tan^{-1}(x) + C.$$

13. $\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$, so the surface area is given by $S = \int_0^a 2\pi y ds = 2\pi \int_0^a \sqrt{1-x^2} \sqrt{1 + \frac{x^2}{1-x^2}} dx$

$$= 2\pi \int_0^a dx = 2\pi a = \frac{2}{3}, \text{ so } a = \frac{1}{3\pi}.$$

14. .

(a) $\lim_{n \rightarrow \infty} \frac{n^2 e^{-3n}}{e^{3n}} = \lim_{n \rightarrow \infty} \frac{n^2}{e^{3n}}$. Apply L'Hospital's Rule to compute the limit of the real-valued function $f(x) = \frac{x^2}{e^{3x}}$.

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}} = \lim_{x \rightarrow \infty} \frac{2x}{3e^{3x}} = \lim_{x \rightarrow \infty} \frac{2}{9e^{3x}} = 0.$$

Therefore, the sequence also approaches 0.

(b) Let $a_n = \frac{n}{3n+2}$. $\lim_{n \rightarrow \infty} a_n = \frac{1}{3} (\neq 0)$, therefore, by the Test for Divergence, the series $\sum_{n=1}^{\infty} \frac{n}{3n+2}$ is divergent.

15. .

(a) $0 \leq \frac{\cos^2(\sqrt{n})}{n(n+2)} \leq \frac{1}{n^2 + 2n} \leq \frac{1}{n^2}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it is a p -series with $p = 2$. Therefore, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{\cos^2(\sqrt{n})}{n(n+2)}$ is convergent.

(b) The terms of the series are all positive. Let $a_n = \frac{1}{1+n^{2/3}}$ and $b_n = \frac{1}{n^{2/3}}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{2/3}}{1+n^{2/3}} = 1$, which means the series $\sum a_n$ and $\sum b_n$ are either both convergent or both divergent. But $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is divergent since it is a p -series with $p = \frac{2}{3}$, therefore

$\sum_{n=1}^{\infty} \frac{1}{1+n^{2/3}}$ is divergent by the Limit Comparison Test.

16. Since the length of C is L , $\int_0^1 \sqrt{(f'(t))^2 + (g'(t))^2} dt = L$. For the new curve, $\frac{dx}{dt} = g'(1-t)(-1)$ and $\frac{dy}{dt} = f'(1-t)(-1)$. Therefore, the length of the new curve is given by $s = \int_0^1 \sqrt{(g'(1-t))^2 + (f'(1-t))^2} dt$. Let $u = 1-t$. Then $du = -dt$, or $dt = -du$. When $t = 0$, $u = 1$, and when $t = 1$, $u = 0$. Therefore, $s = - \int_1^0 \sqrt{(g'(u))^2 + (f'(u))^2} du = \int_0^1 \sqrt{(f'(u))^2 + (g'(u))^2} du = L$.