Fall 2009 Math 152

Exam III Version A Solutions

1. **E** $2\mathbf{a} = \langle 2,4,6\rangle$, $3\mathbf{b} = \langle 9,6,3\rangle$, so $2\mathbf{a} - 3\mathbf{b} = \langle -7,-2,3\rangle$

2. **A** $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-2+3+2}{\sqrt{6}\cdot\sqrt{14}} = \dfrac{3}{\sqrt{84}}$

3. **D** The dot product must be 0, so we have $3x + 4 + 2x = 0$, or $x = -\dfrac{4}{5}$

4. **E** Let **a**, **b**, and **c** be the three vectors respectively. The volume is equal to $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k}$, so the volume is $|(-2)(0) + (0)(1) + (2)(-1)| = 2$

5. **A** Complete the square for each quadratic. $x^2 + (y^2+2y+1) + (z^2+z+\frac{1}{4}) = 1+1+\frac{1}{4} = \frac{9}{4} = r^2$, so $r = \frac{3}{2}$.

6. **C** Series (I) is absolutely convergent since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent P-series. Series (II) is convergent by the Alternating Series Test, but $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a divergent P-Series, so (II) is convergent, but not absolutely convergent.

7. **A** $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Replace x with $-x^2$: $e^{-x^2} = \displaystyle\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$

8. **E** $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Multiply by x^2: $x^2\sin x = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+1)!}$

9. **D** $\cos x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$, so $\cos\pi = -1 = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}$. Multiply by π: $-\pi = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n)!}$.

10. **C** Since the series has a radius of convergence of 3, the series is convergent when $|x-3| < 3$ and is divergent when $|x-3| > 3$. For series (I), $|x-3| = 2(<3)$ and for series (II), $|x-3| = 4(>3)$. Therefore, (I) is convergent and (II) is divergent.

11. **E** Since $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, we must have $a_n \to 0$. Therefore, $\dfrac{1}{1+a_n} \to 1$, so the series is divergent by the Test for Divergence.

12. (i) The vector must also be perpendicular to the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$ which lie in the plane. $\overrightarrow{QP} = \langle 1,-1,-2\rangle$ and $\overrightarrow{QR} = \langle 1,0,0\rangle$. The cross-product is perpendicular to both, so the desired vector is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 1 & 0 & 0 \end{vmatrix} = -2\mathbf{j} + \mathbf{k}$.
(ii) The area of the triangle is one-half the magnitude of the cross-product found above: $\frac{1}{2}\sqrt{(-2)^2+1^2} = \frac{\sqrt{5}}{2}$.

13. (i) The series is an alternating series with $a_n = \dfrac{1}{n\ln n}$, which is positive, decreasing, and approaching zero. Therefore, the series is convergent by the Alternating Series Test.
(ii) To test absolute convergence, look at the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$. Let $f(x) = \dfrac{1}{x\ln x}$. f is positive, continuous, and decreasing. Further, $\displaystyle\int_2^{\infty}\frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\ln(\ln t) - \ln(\ln 2) = \infty$. Therefore, the series is divergent by the Integral Test, so the original series is NOT absolutely convergent.

14. (i) Apply the Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^{n+2}}{\sqrt{2n+3}}\cdot\frac{\sqrt{2n+1}}{2^n(x-1)^{n+1}}\right| = \lim_{n\to\infty}\frac{\sqrt{2n+1}}{\sqrt{2n+3}}\cdot 2|x-1| = 2|x-1|$. For the series to converge, we need $2|x-1| < 1$, or $|x-1| < \frac{1}{2}$. Therefore, the radius of convergence is $\frac{1}{2}$.
(ii) The power series is convergent when $-\frac{1}{2} < x-1 < \frac{1}{2}$, or $\frac{1}{2} < x < \frac{3}{2}$. Test each

endpoint separately. When $x = \frac{1}{2}$, the series becomes $\sum_{n=0}^{\infty} \frac{2^n(\frac{1}{2}-1)^n}{\sqrt{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$, which is convergent by the Alternating Series Test. When $x = \frac{3}{2}$, the series becomes $\sum_{n=0}^{\infty} \frac{2^n(\frac{1}{2})^n}{\sqrt{2n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2n+1}}$, which is divergent by Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Therefore, the interval of convergence is $\left[\frac{1}{2}, \frac{3}{2}\right)$

15. $f(x) = x^5 - 2x^4 + x^3$ (NOTE: This step is not required, but makes the derivatives much easier to compute).
$f(x) = x^5 - 2x^4 + x^3$; $f(1) = 0$
$f'(x) = 5x^4 - 8x^3 + 3x^2$; $f'(1) = 0$
$f''(x) = 20x^3 - 24x^2 + 6x$; $f''(1) = 2$
$f'''(x) = 60x^2 - 48x + 6$; $f'''(1) = 18$
Therefore, $T_3(x) = \frac{2}{2!}(x-1)^2 + \frac{18}{3!}(x-1)^3 = (x-1)^2 + 3(x-1)^3$.

16. (i) $f(x) = \frac{1/16}{1 + x^4/16}$, which is the sum of a Geometric Series with $a = \frac{1}{16}$ and $r = -\frac{x^4}{16}$. Therefore, $f(x) = \sum_{n=0}^{\infty} \frac{1}{16}\left(-\frac{x^4}{16}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{16^{n+1}}$.
(ii) $\int_0^1 f(x)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{16^{n+1}}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)16^{n+1}} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)16^{n+1}}$.