

Math 152 Spring 2010 Exam I Solutions-Form A

1. a. The average value of $f(x) = \frac{x}{(x+1)^2}$ on the interval $[0, 2]$ is defined as

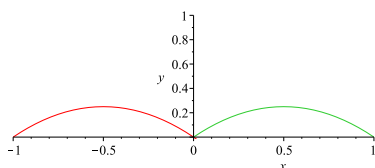
$$f_{ave} = \frac{1}{2} \int_0^2 \frac{x}{(x+1)^2} dx. \text{ Let } u = x + 1. \text{ Then } du = dx, \text{ and } x = u - 1.$$

$$\begin{aligned} f_{ave} &= \frac{1}{2} \int_0^2 \frac{x}{(x+1)^2} dx = \frac{1}{2} \int_1^3 \frac{u-1}{u^2} du \\ &= \frac{1}{2} \int_1^3 \left(\frac{1}{u} - \frac{1}{u^2} \right) du \\ &= \frac{1}{2} \left(\ln u + \frac{1}{u} \right) \Big|_1^3 = \frac{1}{2} \left(\ln 3 - \frac{2}{3} \right). \end{aligned}$$

2. d. Since the cable is 80 feet long and weighs 240 pounds, it weighs 3 pounds per foot. When we pull the first 10 feet to the top, we need to remember we are also pulling the bottom 70 feet of the cable up 10 feet. Thus the total work done is

$$\begin{aligned} W &= \int_0^{10} 3x dx + (70 \text{ feet})(3 \text{ lbs per foot})(10 \text{ feet}) \\ &= \frac{3x^2}{2} \Big|_0^{10} + 2100 = 2250 \text{ foot-pounds.} \end{aligned}$$

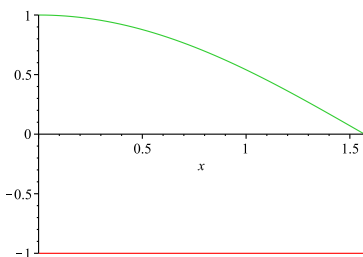
3. b. Revolve the region bounded by $y = x - x^2$, the x -axis, $x = 0$ and $x = 1$ about the y -axis:



We will use the method of cylindrical shells:

$$\begin{aligned} V &= \int_0^1 2\pi x(x - x^2) dx = \int_0^1 2\pi(x^2 - x^3) dx \\ &= 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{\pi}{6}. \end{aligned}$$

4. b. Since the graph of $y = \cos x$ is above the graph of $y = -1$,

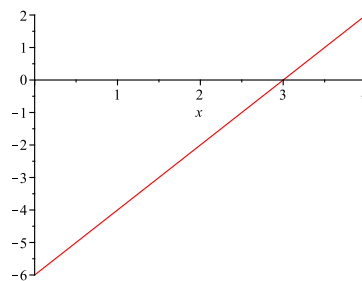


$$\begin{aligned} A &= \int_0^{\pi/2} (\cos x - (-1)) dx \\ &= (\sin x + x) \Big|_0^{\pi/2} = 1 + \frac{\pi}{2} \end{aligned}$$

5. e. To find $\int_0^\pi x \cos\left(\frac{x}{2}\right) dx$, we will use integration by parts where $u = x$ and $dv = \cos\left(\frac{x}{2}\right) dx$. Then $du = dx$ and $v = 2 \sin\left(\frac{x}{2}\right)$.

$$\begin{aligned} \int_0^\pi x \cos\left(\frac{x}{2}\right) dx &= uv - \int v du \\ &= 2x \sin\left(\frac{x}{2}\right) \Big|_0^\pi - \int_0^\pi 2 \sin\left(\frac{x}{2}\right) dx \\ &= \left(2x \sin\left(\frac{x}{2}\right) + 4 \cos\left(\frac{x}{2}\right) \right) \Big|_0^\pi = 2\pi - 4 \end{aligned}$$

6. d. The function $y = 2x - 6$ crosses the x -axis at $x = 3$.



Between $x = 0$ and $x = 3$, the graph of $y = 0$ is above the graph of $y = 2x - 6$. Between $x = 3$ and $x = 4$, the graph of $y = 2x - 6$ is above the graph of $y = 0$. Thus

$$\begin{aligned} A &= \int_0^3 (0 - (2x - 6)) dx + \int_3^4 (2x - 6 - 0) dx \\ &= (-x^2 + 6x) \Big|_0^3 + (x^2 - 6x) \Big|_3^4 = 10. \end{aligned}$$

Note: The enclosed region can also be viewed as the sum of the area of two triangles. The area of the triangle below the x -axis is 9 and the area of the triangle above the x -axis is 1. Hence the sum of these two triangles is 10.

7. e. For $\int \frac{\sec \theta \tan \theta}{4 + \sec \theta} d\theta$, we will do u -substitution where $u = 4 + \sec \theta$. Then $du = \sec \theta \tan \theta d\theta$.

$$\int \frac{\sec \theta \tan \theta}{4 + \sec \theta} d\theta = \int \frac{du}{u} = \ln |u| + C = \ln |4 + \sec \theta| + C$$

8. e. Since the power on cosine is odd, we will factor one $\cos x$ out of $\cos^3 x$:

$$\begin{aligned} & \int (\cos^3 x) (\sqrt{\sin x}) dx \\ &= \int (\cos^2 x) (\sqrt{\sin x} \cos x) dx \\ &= \int ((1 - \sin^2 x)) (\sqrt{\sin x} \cos x) dx. \text{ Now let} \\ & u = \sin x. \text{ Then } du = \cos x dx. \text{ Thus} \\ &= \int ((1 - \sin^2 x)) (\sqrt{\sin x} \cos x) dx \\ &= \int (1 - u^2) (\sqrt{u}) du \\ &= \int (u^{1/2} - u^{5/2}) du \\ &= \frac{2}{3} u^{3/2} - \frac{2}{7} u^{7/2} + C \\ &= \frac{2}{3} (\sin x)^{3/2} - \frac{2}{7} (\sin x)^{7/2} + C \end{aligned}$$

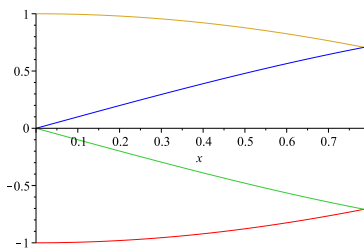
9. c. Using the method of disks, we find that the volume is $V = \int_{-1}^1 \pi(1 - x^2)^2 dx = 2 \int_0^1 \pi(1 - x^2)^2 dx$ (By symmetry).

$$\begin{aligned} V &= 2\pi \int_0^1 (1 - 2x^2 + x^4) dx \\ &= 2\pi \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_0^1 = \frac{16\pi}{15} \end{aligned}$$

10. a. For $f(x) = \frac{1}{\sqrt{x}}$ on $[4, 9]$, $f_{ave} = \frac{1}{5} \int_4^9 \frac{1}{\sqrt{x}} dx = \frac{2}{5} \sqrt{x} \Big|_4^9 = \frac{2}{5}$.

Now, we want to find the value of c so that $f(c) = f_{ave}$. Thus $\frac{1}{\sqrt{c}} = \frac{2}{5}$, hence $c = \frac{25}{4}$.

11. $y = \cos x$ and $y = \sin x$ intersect at $x = \frac{\pi}{4}$. If we revolve this region about the x -axis, we will get a washer cross section:

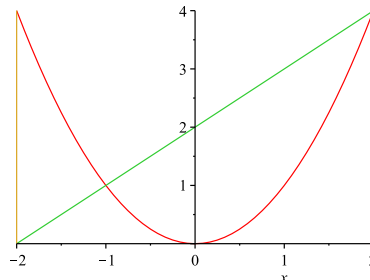


Thus

$$V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) dx$$

$$\begin{aligned} &= \pi \int_0^{\pi/4} \left(\frac{1}{2}(1 + \cos 2x) - \frac{1}{2}(1 - \cos 2x) \right) dx \\ &= \pi \int_0^{\pi/4} \cos 2x dx \\ &= \pi \left(\frac{1}{2} \sin 2x \right) \Big|_0^{\pi/4} = \frac{\pi}{2} \end{aligned}$$

12. The graphs of $y = x^2$ and $y = x + 2$ intersect at $x = -1$ and $x = 2$.



Since $y = x^2$ is above $y = x + 2$ for $-2 < x < -1$ and $y = x + 2$ is above $y = x^2$ for $-1 < x < 2$, we must split the integral at $x = -1$.

$$\begin{aligned} A &= \int_{-2}^{-1} (x^2 - (x + 2)) dx + \int_{-1}^2 (x + 2 - x^2) dx \\ &= \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \Big|_{-2}^{-1} + \left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \right) \Big|_{-1}^2 = \frac{19}{3} \end{aligned}$$

13. We are given that the base of the solid is the ellipse $x^2 + \frac{y^2}{4} = 1$. Since cross sections perpendicular to the x -axis are squares, the volume of a slice is

$V = (2y)^2 dx$. Since $x^2 + \frac{y^2}{4} = 1$, $y^2 = 4 - 4x^2$. Thus $V = 4(4 - 4x^2) dx$. Hence the volume of the solid is

$$V = \int_{-1}^1 (16 - 16x^2) dx = \left(16x - \frac{16x^3}{3} \right) \Big|_{-1}^1 = \frac{64}{3}$$

14. A cross section of water has volume $\pi(4 - y^2)dy$, thus the work done in pumping this slice to the top of the tank is $W = 9800\pi y(4 - y^2)dy$. Hence the total work done in pumping all the water to the top of the tank is

$$\begin{aligned} W &= \int_0^2 9800\pi y(4 - y^2) dy \\ &= 9800\pi (2y^2 - y^4/4) \Big|_0^2 = 39200\pi \text{ Joules.} \end{aligned}$$

$$\begin{aligned}
15. \text{ a.) } & \int (\tan^3 x) (\sec^5 x) dx \\
&= \int (\tan^2 x) ((\sec^4 x)(\sec x \tan x)) dx \\
&= \int (\sec^2 x - 1) ((\sec^4 x)(\sec x \tan x)) dx \\
&\text{Let } u = \sec x. \text{ Then } du = \sec x \tan x. \text{ Thus} \\
&= \int (\sec^2 x - 1) ((\sec^4 x)(\sec x \tan x)) dx \\
&= \int (u^2 - 1)u^4 du = \int (u^6 - u^2) du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \frac{(\sec x)^7}{7} - \frac{(\sec x)^5}{5} + C
\end{aligned}$$

b.) To evaluate $\int x^3 e^{x^2} dx$, we will first do a t -substitution. Let $t = x^2$. Then $dt = 2x dx$.

$$\begin{aligned}
\int x^3 e^{x^2} dx &= \int x x^2 e^{x^2} dx \\
&= \int \frac{1}{2} t e^t dt.
\end{aligned}$$

Using integration by parts, let $u = t$ and $dv = e^t dt$. Then $du = dt$ and $v = e^t$.

$$\begin{aligned}
&= \int \frac{1}{2} t e^t dt = \frac{1}{2} \left(uv - \int v du \right) \\
&= \frac{1}{2} \left(t e^t - \int e^t dt \right) \\
&= \frac{1}{2} (t e^t - e^t) + C \\
&= \frac{1}{2} (x^2 e^{x^2} - e^{x^2}) + C
\end{aligned}$$