

## Math 152 Spring 2010 Exam II Solutions-Form A

1. B. To find the length of the curve  $y = 4x^{3/2}$ ,  $0 \leq x \leq 1$ , we will use the formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ where } a = 0, b = 1 \text{ and } \frac{dy}{dx} = 6x^{1/2}.$$

$$L = \int_0^1 \sqrt{1 + (6\sqrt{x})^2} dx = \int_0^1 \sqrt{1 + 36x} dx. \text{ Using a } u \text{ substitution with } u = 1 + 36x, \text{ we obtain}$$

$$\begin{aligned} \int_0^1 \sqrt{1 + 36x} dx &= \int_1^{37} \frac{1}{36} \sqrt{u} du \\ &= \frac{1}{36} \frac{2}{3} u^{3/2} \Big|_1^{37} = \frac{1}{54} (37\sqrt{37} - 1) \end{aligned}$$

2. D. To find the surface area obtained by rotating the curve  $y = e^x + x$ ,  $0 \leq x \leq 1$ , about the  $y$  axis, we will use the formula

$$SA = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \text{ where } a = 0, b = 1 \text{ and } \frac{dy}{dx} = e^x + 1.$$

$$\begin{aligned} SA &= \int_0^1 2\pi x \sqrt{1 + (e^x + 1)^2} dx \\ &= \int_0^1 2\pi x \sqrt{e^{2x} + 2e^x + 2} dx \end{aligned}$$

3. E: To find  $\int \frac{x^2}{x+3} dx$ , we must first perform long division since the power in the denominator is not higher than the power in the numerator.

$$\begin{array}{r} x - 3 \\ x + 3 \overline{) x^2} \\ \underline{-x^2 - 3x} \phantom{0} \\ -3x \phantom{0} \\ \underline{3x + 9} \\ 9 \end{array}$$

$$\text{Hence } \frac{x^2}{x+3} = x - 3 + \frac{9}{x+3}.$$

$$\begin{aligned} \int \frac{x^2}{x+3} dx &= \int \left( x - 3 + \frac{9}{x+3} \right) dx \\ &= \frac{x^2}{2} - 3x + 9 \ln|x+3| + C \end{aligned}$$

$$\begin{aligned} 4. \text{ A: } \int_0^\infty e^{-2x} dx &= \lim_{t \rightarrow \infty} \int_0^t e^{-2x} dx \\ &= \lim_{t \rightarrow \infty} \left. -\frac{1}{2} e^{-2x} \right|_0^t \\ &= \lim_{t \rightarrow \infty} \frac{-1}{2} (e^{-2t} - 1) = \frac{1}{2} \end{aligned}$$

$$5. \text{ C: } \lim_{n \rightarrow \infty} \sqrt{\frac{3n+1}{4n+3}} = \sqrt{\lim_{n \rightarrow \infty} \frac{3n+1}{4n+3}} = \sqrt{\frac{3}{4}}$$

6. E: First note that in order for  $\sum_{n=1}^\infty \frac{n^2+1}{n^p+n}$  to converge, we must have  $\lim_{n \rightarrow \infty} \frac{n^2+1}{n^p+n} = 0$ . Thus we may assume that  $p > 2$ . Now, to determine all values of  $p$  for which the series  $\sum_{n=1}^\infty \frac{n^2+1}{n^p+n}$  converges, we will do the Limit Comparison Test with  $a_n = \frac{n^2+1}{n^p+n}$  and  $b_n = \frac{n^2}{n^p}$ .

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n^2+1}{n^p+n}}{\frac{n^2}{n^p}} = \lim_{n \rightarrow \infty} \frac{n^{2+p} + n^p}{n^{2+p} + n^3} = 1.$$

Thus both series converge or both series diverge. Now,  $\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty \frac{n^2}{n^p}$  which will converge by  $p$ -series if  $p$  is greater than 3. Hence,  $\sum_{n=1}^\infty \frac{n^2+1}{n^p+n}$  converges only if  $p > 3$ .

7. A. To find the surface area obtained by rotating the curve  $x = \sin(2t)$ ,  $y = \cos(2t)$ ,  $0 \leq t \leq \frac{\pi}{4}$ , about the  $x$ -axis, we will use the formula

$$SA = 2\pi \int_\alpha^\beta y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt, \text{ where}$$

$$y(t) = \cos(2t), \alpha = 0 \text{ and } \beta = \frac{\pi}{4}. \text{ Now,}$$

$$\frac{dx}{dt} = 2 \cos(2t) \text{ and } \frac{dy}{dt} = -2 \sin(2t).$$

$$\begin{aligned} SA &= 2\pi \int_0^{\pi/4} \cos(2t) \sqrt{(2 \cos(2t))^2 + (-2 \sin(2t))^2} dt \\ &= 2\pi \int_0^{\pi/4} \cos(2t) \sqrt{4(\cos^2(2t) + \sin^2(2t))} dt \\ &= 2\pi \int_0^{\pi/4} \cos(2t) \sqrt{4} dt \\ &= 4\pi \int_0^{\pi/4} \cos(2t) dt \\ &= 2\pi \sin(2t) \Big|_0^{\pi/4} = 2\pi \end{aligned}$$

8. A: The third partial sum of the series  $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$  is

$$s_3 = \sum_{i=1}^3 \frac{\cos(i\pi)}{i} = -1 + \frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$$

9. D. Using the trigonometric substitution

$x = 2 \sin(\theta)$ , we find

$$\begin{aligned} \int_0^1 \sqrt{4-x^2} dx &= \int_0^{\pi/6} \sqrt{4-4\sin^2\theta} 2 \cos(\theta) d\theta \\ &= 4 \int_0^{\pi/6} \cos^2 \theta d\theta \end{aligned}$$

10. A: Since  $\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx \leq \int_0^1 \frac{1}{\sqrt{x}} dx$ , and

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{t \rightarrow 0^+} \int_t^1 x^{-1/2} dx \\ &= \lim_{t \rightarrow 0^+} 2\sqrt{x} \Big|_t^1 = \lim_{t \rightarrow 0^+} (2 - 2\sqrt{t}) = 2. \end{aligned}$$

Since the larger integral converges, so does the smaller integral by the comparison test.

11. To find  $\int \frac{x+8}{x^3+4x} dx$ , we must find the partial fraction decomposition of  $\frac{x+8}{x^3+4x}$ :

$$\frac{x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$x+8 = A(x^2+4) + (Bx+C)(x).$$

$x+8 = (A+B)x^2 + Cx + 4A$ , hence by equating coefficients,  $A+B=0$ ,  $C=1$  and  $4A=8$ . This yields  $A=2$ ,  $B=-2$  and  $C=1$ .

$$\begin{aligned} \int \frac{x+8}{x^3+4x} dx &= \int \left( \frac{2}{x} + \frac{-2x+1}{x^2+4} \right) dx \\ &= \int \left( \frac{2}{x} + \frac{-2x}{x^2+4} + \frac{1}{x^2+4} \right) dx \\ &= 2 \ln|x| - \ln(x^2+4) + \frac{1}{2} \arctan \frac{x}{2} + C \end{aligned}$$

12. To find  $\int \frac{x^3}{\sqrt{x^2+1}} dx$ , we will use the trigonometric substitution  $x = \tan \theta$ . Then  $dx = \sec^2 \theta d\theta$ . Thus

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2+1}} dx &= \int \frac{\tan^3(\theta)}{\sqrt{\tan^2(\theta)+1}} \sec^2(\theta) d\theta \\ &= \int \tan^3(\theta) \sec(\theta) d\theta \\ &= \int \tan^2(\theta) \sec(\theta) \tan(\theta) d\theta \\ &= \int (\sec^2(\theta)-1) \sec(\theta) \tan(\theta) d\theta. \end{aligned}$$

Let  $u = \sec(\theta)$ . Then  $du = \sec(\theta) \tan(\theta) d\theta$

$$\begin{aligned} \int (\sec^2(\theta)-1) \sec(\theta) \tan(\theta) d\theta &= \int (u^2-1) du \\ &= \frac{u^3}{3} - u + C = \frac{1}{3} \sec^3(\theta) - \sec(\theta) + C \end{aligned}$$

Now, since  $x = \tan \theta$ ,  $\sec(\theta) = \sqrt{x^2+1}$ . Substitute this into  $\frac{1}{3} \sec^3(\theta) - \sec(\theta) + C$  gives us

$$\frac{1}{3} (\sqrt{x^2+1})^3 - \sqrt{x^2+1} + C$$

13. a.)  $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n\sqrt[4]{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$ , and  $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$  is a convergent  $p$  series ( $p = \frac{5}{4}$ ). Thus, by the comparison test, since the larger series converges, so does the smaller.

b.)  $\sum_{n=1}^{\infty} \ln \frac{n}{2n+4}$  diverges by the Test for Divergence, since  $\lim_{n \rightarrow \infty} \ln \frac{n}{2n+4} = \ln\left(\frac{1}{2}\right) \neq 0$

14.  $\sum_{n=1}^{\infty} \frac{2^n + (-4)^n}{6^n} = \sum_{n=1}^{\infty} \left( \frac{2^n}{6^n} + \frac{(-4)^n}{6^n} \right)$

$\sum_{n=1}^{\infty} \frac{2^n}{6^n} + \sum_{n=1}^{\infty} \frac{(-4)^n}{6^n}$   
 $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$ . Both of these are convergent geometric series since  $-1 < r < 1$  in both cases. To find the sum:

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n \\ &= \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{-2}{3} \left(\frac{-2}{3}\right)^{n-1} \\ &= \frac{1/3}{1-1/3} + \frac{-2/3}{1+2/3} = \frac{1}{10} \end{aligned}$$

15. a.) Use the integral test to prove  $\sum_{n=1}^{\infty} ne^{-n^2}$  converges. Let  $f(x) = xe^{-x^2}$ . This is a positive, continuous, decreasing function on  $[1, \infty)$ . Now

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{t \rightarrow \infty} \int_1^t xe^{-x^2} dx.$$

Using  $u$ -substitution with  $u = -x^2$ , we find that

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_1^t xe^{-x^2} dx &= \lim_{t \rightarrow \infty} -\frac{1}{2} e^{-x^2} \Big|_1^t \\ &= \lim_{t \rightarrow \infty} -\frac{1}{2} (e^{-t^2} - e^{-1}) = \frac{1}{2e}. \end{aligned}$$

Hence since the integral converges, so does the series by the integral test.

b.)  $s_4 = \sum_{i=1}^4 ie^{-i^2} = e^{-1} + 2e^{-4} + 3e^{-9} + 4e^{-16}$ . To get an upper bound on the error, we recall that the remainder,  $R_4$ , is bounded above:

$$R_4 < \int_4^{\infty} xe^{-x^2} dx = \lim_{t \rightarrow \infty} -\frac{1}{2}e^{-x^2} \Big|_4^t = \frac{1}{2e^{16}}.$$