Math 152 Spring 2010 Exam II Solutions-Form B

1. D: $\int_0^\infty e^{-2x}\,dx = \lim_{t\to\infty}\int_0^t e^{-2x}\,dx$
$= \lim_{t\to\infty} -\frac{1}{2}e^{-2x}\Big|_0^t$
$= \lim_{t\to\infty} \frac{-1}{2}\left(e^{-2t}-1\right) = \frac{1}{2}$

2. E. To find the surface area obtained by rotating the curve $x=\sin(2t)$, $y=\cos(2t)$, $0\le t\le \frac{\pi}{4}$, about the x-axis, we will use the formula
$SA = 2\pi\int_\alpha^\beta y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$, where
$y(t)=\cos(2t)$, $\alpha=0$ and $\beta=\frac{\pi}{4}$. Now,
$\frac{dx}{dt}=2\cos(2t)$ and $\frac{dy}{dt}=-2\sin(2t)$.
$SA = 2\pi\int_0^{\pi/4}\cos(2t)\sqrt{(2\cos(2t))^2+(-2\sin(2t))^2}\,dt$
$= 2\pi\int_0^{\pi/4}\cos(2t)\sqrt{4\left(\cos^2(2t)+\sin^2(2t)\right)}\,dt$
$= 2\pi\int_0^{\pi/4}\cos(2t)\sqrt{4}\,dt$
$= 4\pi\int_0^{\pi/4}\cos(2t)\,dt$
$= 2\pi\sin(2t)|_0^{\pi/4} = 2\pi$

3. A: To find $\int\frac{x^2}{x+3}\,dx$, we must first perform long division since the power in the denominator is not higher than the power in the numerator.

$$\begin{array}{r|l} & x-3 \\ \hline x+3) & x^2 \\ & -x^2-3x \\ \hline & -3x \\ & 3x+9 \\ \hline & 9 \end{array}$$

Hence $\frac{x^2}{x+3} = x-3+\frac{9}{x+3}$.
$\int\frac{x^2}{x+3}\,dx = \int\left(x-3+\frac{9}{x+3}\right)dx$
$= \frac{x^2}{2}-3x+9\ln|x+3|+C$

4. A: First note that in order for $\sum_{n=1}^\infty\frac{n^2+1}{n^p+n}$ to converge, we must have $\lim_{n\to\infty}\frac{n^2+1}{n^p+n}=0$. Thus we may assume that $p>2$. Now, to determine all values of p for which the series $\sum_{n=1}^\infty\frac{n^2+1}{n^p+n}$ converges, we will do the Limit Comparison Test with $a_n=\frac{n^2+1}{n^p+n}$ and $b_n=\frac{n^2}{n^p}$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n^2+1}{n^p+n}}{\frac{n^2}{n^p}} = \lim_{n\to\infty}\frac{n^{2+p}+n^p}{n^{2+p}+n^3} = 1.$$

Thus both series converge or both series diverge. Now, $\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty\frac{n^2}{n^p}$ which will converge by p-series if p is greater then 3. Hence, $\sum_{n=1}^\infty\frac{n^2+1}{n^p+n}$ converges only if $p>3$.

5. D: The third partial sum of the series $\sum_{n=1}^\infty\frac{\cos(n\pi)}{n}$ is
$s_3=\sum_{i=1}^3\frac{\cos(i\pi)}{i} = -1+\frac{1}{2}-\frac{1}{3} = -\frac{5}{6}$

6. C. To find the length of the curve $y=4x^{3/2}$, $0\le x\le 1$, we will use the formula
$L = \int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ where $a=0$, $b=1$ and $\frac{dy}{dx}=6x^{1/2}$.
$L=\int_0^1\sqrt{1+(6\sqrt{x})^2}\,dx = \int_0^1\sqrt{1+36x}\,dx$. Using a u substitution with $u=1+36x$, we obtain
$\int_0^1\sqrt{1+36x}\,dx = \int_1^{37}\frac{1}{36}\sqrt{u}\,du$
$= \frac{1}{36}\frac{2}{3}u^{3/2}\Big|_1^{37} = \frac{1}{54}(37\sqrt{37}-1)$

7. B: Since $\int_0^1\frac{e^{-x}}{\sqrt{x}}\,dx \le \int_0^1\frac{1}{\sqrt{x}}\,dx$, and
$\int_0^1\frac{1}{\sqrt{x}}\,dx = \lim_{t\to0^+}\int_t^1 x^{-1/2}\,dx$
$= \lim_{t\to0^+}2\sqrt{x}|_t^1 = \lim_{t\to0^+}\left(2-2\sqrt{t}\right) = 2$. Since the larger integral converges, so does the smaller integral by the comparison test.

8. B. Using the trigonometric substitution $x = 2\sin(\theta)$, we find

$$\int_0^1 \sqrt{4-x^2}\,dx = \int_0^{\pi/6} \sqrt{4-4\sin^2\theta}\,2\cos(\theta)d\theta$$

$$= 4\int_0^{\pi/6} \cos^2\theta\,d\theta$$

9. A. To find the surface area obtained by rotating the curve $y = e^x + x$, $0 \le x \le 1$, about the y axis, we will use the formula

$SA = \int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, where $a = 0$, $b = 1$ and $\frac{dy}{dx} = e^x + 1$.

$$SA = \int_0^1 2\pi x\sqrt{1+(e^x+1)^2}\,dx$$

$$= \int_0^1 2\pi x\sqrt{e^{2x}+2e^x+2}\,dx$$

10. B: $\lim_{n\to\infty}\sqrt{\frac{3n+1}{4n+3}} = \sqrt{\lim_{n\to\infty}\frac{3n+1}{4n+3}} = \sqrt{\frac{3}{4}}$

11. To find $\int \frac{x+8}{x^3+4x}\,dx$, we must find the partial fraction decomposition of $\frac{x+8}{x^3+4x}$:

$$\frac{x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$x + 8 = A(x^2+4) + (Bx+C)(x)$.

$x + 8 = (A+B)x^2 + Cx + 4A$, hence by equating coefficients, $A + B = 0$, $C = 1$ and $4A = 8$. This yields $A = 2$, $B = -2$ and $C = 1$.

$$\int \frac{x+8}{x^3+4x}\,dx = \int \left(\frac{2}{x} + \frac{-2x+1}{x^2+4}\right)\,dx$$

$$= \int \left(\frac{2}{x} + \frac{-2x}{x^2+4} + \frac{1}{x^2+4}\right)\,dx$$

$$= 2\ln|x| - \ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C$$

12. To find $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$, we will use the trigonometric substitution $x = \tan\theta$. Then $dx = \sec^2\theta d\theta$. Thus

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{\tan^3(\theta)}{\sqrt{\tan^2(\theta)+1}}\sec^2(\theta)d\theta$$

$$= \int \tan^3(\theta)\sec(\theta)d\theta$$

$$= \int \tan^2(\theta)\sec(\theta)\tan(\theta)d\theta$$

$= \int (\sec^2(\theta)-1)\sec(\theta)\tan(\theta)d\theta$. Let $u = \sec(\theta)$. Then $du = \sec(\theta)\tan(\theta)d\theta$

$$\int (\sec^2(\theta)-1)\sec(\theta)\tan(\theta)d\theta = \int (u^2-1)\,du$$

$$= \frac{u^3}{3} - u + C = \frac{1}{3}\sec^3(\theta) - \sec(\theta) + C$$

Now, since $x = \tan\theta$, $\sec(\theta) = \sqrt{x^2+1}$. Substitute this into $\frac{1}{3}\sec^3(\theta) - \sec(\theta) + C$ gives us

$$\frac{1}{3}(\sqrt{x^2+1})^3 - \sqrt{x^2+1} + C$$

13. a.) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n\sqrt[4]{n}} \le \sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$, and $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$ is a convergent p series ($p = \frac{5}{4}$). Thus, by the comparison test, since the larger series converges, so does the smaller.

b.) $\sum_{n=1}^{\infty} \ln\frac{n}{2n+4}$ diverges by the Test for Divergence, since $\lim_{n\to\infty} \ln\frac{n}{2n+4} = \ln(\frac{1}{2}) \neq 0$

14. $\sum_{n=1}^{\infty} \frac{2^n + (-4)^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{2^n}{6^n} + \frac{(-4)^n}{6^n}\right)$

$$\sum_{n=1}^{\infty} \frac{2^n}{6^n} + \sum_{n=1}^{\infty} \frac{(-4)^n}{6^n}$$

$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n$. Both of these are convergent geometric series since $-1 < r < 1$ in both cases. To find the sum:

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{-2}{3}\left(\frac{-2}{3}\right)^{n-1}$$

$$= \frac{1/3}{1-1/3} + \frac{-2/3}{1+2/3} = \frac{1}{10}$$

15. a.) Use the integral test to prove $\sum_{n=1}^{\infty} ne^{-n^2}$ converges. Let $f(x) = xe^{-x^2}$. This is a positive, continuous, decreasing function on $[1,\infty)$. Now

$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx$. Using u-substitution with $u = -x^2$, we find that

$$\lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^t$$

$= \lim_{t\to\infty} -\frac{1}{2}\left(e^{-t^2} - e^{-1}\right) = \frac{1}{2e}$. Hence since the integral converges, so does the series by the integral test.

b.) $s_4 = \sum_{i=1}^{4} ie^{-i^2} = e^{-1} + 2e^{-4} + 3e^{-9} + 4e^{-16}$. To get an upper bound on the error, we recall that the remainder, R_4, is bounded above:

$$R_4 < \int_4^\infty xe^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\bigg|_4^t = \frac{1}{2e^{16}}.$$