

Math 152 Spring 2010 Exam III Solutions-Form A

1. E: To find the intersection of the sphere $(x+1)^2 + (y-2)^2 + (z-3)^2 = 25$ with the xz -plane, we set $y = 0$ in the equation
- $$(x+1)^2 + (y-2)^2 + (z-3)^2 = 25:$$
- $$(x+1)^2 + (0-2)^2 + (z-3)^2 = 25 \text{ yields}$$
- $$(x+1)^2 + (z-3)^2 = 21$$

2. A: To find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(x+1)^n (2n+1)!}{10^{2n}}$, we need to determine what values of x satisfies

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1.$$

$$\lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1} (2n+3)!}{10^{2n+1} (n+1)!} \cdot \frac{10^{2n}}{(x+1)^n (2n+1)!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(x+1)(2n+3)(2n+2)}{10(n+1)} \right| < 1 \text{ only if}$$

$x = -1$ (otherwise if $x \neq -1$, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$). Hence this series will only converge if $x = -1$, yielding an interval of convergence of $I = \{-1\}$.

3. D: Which of the following series converge absolutely? Recall that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

(a): $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges by p series.

(b): $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$ which is a divergent p -series.

(c): $\sum_{n=1}^{\infty} |(-1)^n| = \sum_{n=1}^{\infty} (1)$ which diverges by the Test for Divergence.

(d): $\sum_{n=1}^{\infty} \left| \frac{\cos(n\pi)}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ which is a convergent p series. Hence this is the only series that converges absolutely.

4. B: To find a power series representation for $\arctan(x^3)$, we will differentiate first:

$$\frac{d}{dx}(\arctan(x^3)) = \frac{3x^2}{1+x^6}. \text{ Thus}$$

$$\arctan(x^3) = \int \frac{3x^2}{1+x^6} dx$$

$$\arctan(x^3) = \int 3x^2 \frac{1}{1-(-x^6)} dx$$

$$\arctan(x^3) = \int 3x^2 \sum_{n=0}^{\infty} (-x^6)^n dx \text{ where } |x| < 1.$$

$$\arctan(x^3) = \int 3x^2 \sum_{n=0}^{\infty} (-1)^n x^{6n} dx$$

$$\arctan(x^3) = \int 3 \sum_{n=0}^{\infty} (-1)^n x^{6n+2} dx$$

$$\arctan(x^3) = C + 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{6n+3}$$

Now we find $C = 0$ by substituting $x = 0$ in both sides of the equation above. Hence,

$$\arctan(x^3) = 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{6n+3}$$

$$\arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$

Note: A much more direct way to work this problem is to recall that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, in which case, by substitution,

$$\arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$

5. A: To find the unit vector in the direction of $\mathbf{b} - \mathbf{a}$ where $\mathbf{a} = \langle 0, 2, 1 \rangle$ and $\mathbf{b} = \langle 1, 1, 3 \rangle$, we will first find

$\mathbf{b} - \mathbf{a} = \langle 1, 1, 3 \rangle - \langle 0, 2, 1 \rangle = \langle 1, -1, 2 \rangle$. To make this a unit vector, we will divide by the magnitude:

$$\frac{\langle 1, -1, 2 \rangle}{|\langle 1, -1, 2 \rangle|} = \left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle.$$

6. A: Given the triangle with vertices $A(2, -2, 5)$, $B(1, 1, 4)$ and $C(3, 1, 3)$, find the cosine of the angle at B .

Let β be the angle at B . Then $\cos \beta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|}$,

where

$$\overrightarrow{BA} = \langle 1, -3, 1 \rangle \text{ and } \overrightarrow{BC} = \langle 2, 0, -1 \rangle.$$

$$\cos \beta = \frac{\langle 1, -3, 1 \rangle \cdot \langle 2, 0, -1 \rangle}{|\langle 1, -3, 1 \rangle| |\langle 2, 0, -1 \rangle|} = \frac{1}{\sqrt{55}}$$

7. C: Recall the Maclaurin series for e^x :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \text{ Thus}$$

$$\frac{e^x - 1 - x}{x^2} = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots - 1 - x}{x^2}$$

$$= \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}{x^2}$$

$$= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}$$

8. B: To represent $\frac{1}{9+4x^2}$ as a power series, note that

$$\frac{1}{9+4x^2} = \frac{1}{9 \left(1 + \frac{4x^2}{9}\right)} = \frac{1}{9} \frac{1}{1 - \left(-\frac{4x^2}{9}\right)}$$

$$= \frac{1}{9} \sum_{n=0}^{\infty} \left(\frac{-4x^2}{9}\right)^n, \text{ where } \left|\frac{-4x^2}{9}\right| < 1.$$

Thus $|x^2| < \frac{9}{4}$, hence $|x| < \frac{3}{2}$. Thus the radius of convergence is $R = \frac{3}{2}$

9. D: Recall The Alternating Series Estimation Theorem states that if $\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n > 0$, is a convergent alternating series, and we used a partial sum $s_n = \sum_{i=1}^n (-1)^i a_i$ to approximate the sum, then an upper bound on the absolute value of the remainder is $|R_n| \leq a_{n+1}$. To determine how many terms of the series do we need to add in order to find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with error

less than $\frac{1}{60}$, we need to solve $a_{n+1} < \frac{1}{60}$. Thus $\frac{1}{(n+1)^2} < \frac{1}{60}$. The smallest value of n for which this inequality holds is $n = 7$.

10. D: The series $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^2 + 1}$ diverges by the Test for Divergence since $\lim_{n \rightarrow \infty} \frac{(-1)^n n^2}{n^2 + 1} \neq 0$.
11. To find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-3)^n (2x-1)^n}{n}$,

use the Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1} (2x-1)^{n+1}}{n+1} \frac{n}{(-3)^n (2x-1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{-3(2x-1)}{n+1} n \right|$$

$$= |-3(2x-1)|$$

$$= |-3||2x-1|$$

$$= 3|2(x - \frac{1}{2})|$$

$$= 6|x - \frac{1}{2}|. \text{ Now the Ratio Test says if}$$

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

Hence

$6|x - \frac{1}{2}| < 1$, so $|x - \frac{1}{2}| < \frac{1}{6}$, thus $R = \frac{1}{6}$. Now test the endpoints of the interval:

$|x - \frac{1}{2}| < \frac{1}{6}$ yields $\frac{1}{3} < x < \frac{2}{3}$. Testing $x = \frac{1}{3}$:

$\sum_{n=1}^{\infty} \frac{(-3)^n (-1/3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by p -series.

Testing $x = \frac{2}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n (1/3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

which converges by the Alternating Series Test.

Thus the interval of convergence is $\frac{1}{3} < x \leq \frac{2}{3}$.

12. (i) To find a Maclaurin Series representaton for

$$f(x) = \sin\left(\frac{x^2}{3}\right), \text{ recall that}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \text{ Thus}$$

$$f(x) = \sin\left(\frac{x^2}{3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^2}{3}\right)^{2n+1}}{(2n+1)!}.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{3^{2n+1}(2n+1)!}.$$

(ii) $\int_0^1 \sin\left(\frac{x^2}{3}\right) dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{3^{2n+1}(2n+1)!} dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{3^{2n+1}(4n+3)(2n+1)!} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(4n+3)(2n+1)!}$$

- (iii) The sum of the first three nonzero terms is

$$s_2 = \frac{1}{3(3)} - \frac{1}{3^3(3!)(7)} + \frac{1}{3^5(5!)(11)}.$$

$$|R_2| < \frac{1}{3^7 7! (15)}$$

13. To determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^4}$ converges absolutely, we will consider the series

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^4} \right|$$

$$= \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}. \text{ We will use the integral test with}$$

$$f(x) = \frac{1}{x(\ln x)^4}.$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^4} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^4} dx. \text{ Using } u\text{-}$$

substitution with $u = \ln x$, we find

$$\lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^4} dx = \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} \frac{du}{u^4}$$

$$= \lim_{t \rightarrow \infty} \left. -\frac{1}{3u^3} \right|_{\ln 2}^{\ln t}$$

$$= \frac{1}{3(\ln 2)^3}. \text{ Thus since } \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^4} \right| \text{ converges,}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^4} \text{ converges absolutely.}$$

14. The Taylor series for $f(x) = \ln x$ at $a = 4$ is

$$\ln x = \sum_{n=0}^{\infty} \frac{f^n(4)}{n!} (x-4)^n. \text{ We must find a formula}$$

for $f^n(4)$. To do this, we will try to find a pattern for the derivatives:

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f^3(x) = \frac{2}{x^3}$$

$$f^4(x) = -\frac{3 * 2}{x^4}$$

$$f^5(x) = \frac{4 * 3 * 2}{x^5}$$

From this pattern, we can see that

$$f^n(x) = (-1)^{n+1} \frac{(n-1)!}{x^n} \text{ for } n > 0.$$

$$\text{Thus } f^n(4) = (-1)^{n+1} \frac{(n-1)!}{4^n} \text{ for } n > 0.$$

$$\ln x = \sum_{n=0}^{\infty} \frac{f^n(4)}{n!} (x-4)^n$$

$$= f(4) + \sum_{n=1}^{\infty} \frac{f^n(4)}{n!} (x-4)^n$$

$$= \ln(4) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{4^n n!} (x-4)^n$$

$$= \ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 4^n} (x-4)^n$$

Note: Another acceptable way to work this problem is to note that

$$\ln(x) = \ln(4 + (x-4)) = \ln(4) + \ln(1 + (x-4)/4)$$

and then to use the Taylor series for $\ln(1+u)$ about $a = 0$ with $u = (x-4)/4$

15. (i) The fourth degree Taylor polynomial for $f(x)$ centered at $a = 2$ is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3 + \frac{f^{(4)}(2)}{24}(x-2)^4.$$

$$= 1 - (x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{24}(x-2)^4$$

- (ii) Now, by Taylor's inequality,

$$|R_4(x)| < \frac{M}{5!} |x-2|^5, \text{ where } M = \max |f^{(5)}(x)| \text{ for } -1 \leq x \leq 5. \text{ Now, } |f^{(5)}(x)| = |-e^{2-x}| \text{ which takes on a maximum when } x = -1. \text{ Therefore } M = e^3.$$

$$|R_4(x)| < \frac{e^3}{5!} |x-2|^5 \leq \frac{e^3}{5!} (3)^5.$$