Math 152 Spring 2010 Exam III Solutions-Form B

1. A: The series $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^2+1}$ diverges by the Test for Divergence since $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1} \neq 0$.

2. A: Recall the Maclaurin series for e^x:

   $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

   $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$ Thus

   $$\frac{e^x - 1 - x}{x^2} = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... - 1 - x}{x^2}$$

   $$= \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...}{x^2}$$

   $$= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + ...$$

   $$= \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}$$

3. B: To find the unit vector in the direction of $\mathbf{b} - \mathbf{a}$ where $\mathbf{a} = \langle 0, 2, 1 \rangle$ and $\mathbf{b} = \langle 1, 1, 3 \rangle$, we will first find

   $\mathbf{b} - \mathbf{a} = \langle 1, 1, 3 \rangle - \langle 0, 2, 1 \rangle = \langle 1, -1, 2 \rangle$. To make this a unit vector, we will divide by the magnitude:

   $$\frac{\langle 1, -1, 2 \rangle}{|\langle 1, -1, 2 \rangle|} = \left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle.$$

4. D: Recall The Alternating Series Estimation Theorem states that if $\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n > 0$, is a convergent alternating series, and we used a partial sum $s_n = \sum_{i=1}^{n} (-1)^i a_i$ to approximate the sum, then an upper bound on the absolute value of the remainder is $|R_n| \leq a_{n+1}$. To determine how many terms of the series do we need to add in order to find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with error less than $\frac{1}{60}$, we need to solve $a_{n+1} < \frac{1}{60}$. Thus $\frac{1}{(n+1)^2} < \frac{1}{60}$. The smallest value of n for which this inequality holds is $n = 7$.

5. B: Which of the following series converge absolutely? Recall that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

   (a): $\sum_{n=1}^{\infty} |(-1)^n| = \sum_{n=1}^{\infty} (1)$ which diverges by the Test for Divergence.

   (b): $\sum_{n=1}^{\infty} \left| \frac{\cos(n\pi)}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ which is a convergent p series. Hence this is the only series that converges absolutely.

   (c): $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges by p series.

   (d) $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$ which is a divergent p-series.

6. A: To find the intersection of the sphere $(x+1)^2 + (y-2)^2 + (z-3)^2 = 25$ with the xz-plane, we set $y = 0$ in the equation

   $(x+1)^2 + (y-2)^2 + (z-3)^2 = 25$:

   $(x+1)^2 + (0-2)^2 + (z-3)^2 = 25$ yields

   $(x+1)^2 + (z-3)^2 = 21$

7. E: To find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(x+1)^n (2n+1)!}{10^n n!}$, we need to determine what values of x satisfies

   $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$.

   $$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}(2n+3)!}{10^{n+1}(n+1)!} \frac{10^n n!}{(x+1)^n (2n+1)!} \right|$$

   $$= \lim_{n\to\infty} \left| \frac{(x+1)(2n+3)(2n+2)}{10(n+1)} \right| < 1 \text{ only if}$$

   $x = -1$ (otherwise if $x \neq -1$, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$). Hence this series will only converge if $x = -1$, yielding an interval of convergence of $I = \{-1\}$.

8. B: Given the triangle with vertices $A(2,-2,5)$, $B(1,1,4)$ and $C(3,1,3)$, find the cosine of the angle at B.

Let β be the angle at B. Then $\cos\beta = \dfrac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$, where

$\overrightarrow{BA} = \langle 1,-3,1\rangle$ and $\overrightarrow{BC} = \langle 2,0,-1\rangle$.

$$\cos\beta = \frac{\langle 1,-3,1\rangle\cdot\langle 2,0,-1\rangle}{|\langle 1,-3,1\rangle||\langle 2,0,-1\rangle|} = \frac{1}{\sqrt{55}}$$

9. C: To find a power series representation for $\arctan(x^3)$, we will differentiate first:

$\dfrac{d}{dx}(\arctan(x^3)) = \dfrac{3x^2}{1+x^6}$. Thus

$$\arctan(x^3) = \int \frac{3x^2}{1+x^6}\,dx$$

$$\arctan(x^3) = \int 3x^2\frac{1}{1-(-x^6)}\,dx$$

$$\arctan(x^3) = \int 3x^2\sum_{n=0}^{\infty}(-x^6)^n\,dx \text{ where } |x|<1.$$

$$\arctan(x^3) = \int 3x^2\sum_{n=0}^{\infty}(-1)^n x^{6n}\,dx$$

$$\arctan(x^3) = \int 3\sum_{n=0}^{\infty}(-1)^n x^{6n+2}\,dx$$

$$\arctan(x^3) = C + 3\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{6n+3}$$

Now we find $C=0$ by substituting $x=0$ in both sides of the equation above. Hence,

$$\arctan(x^3) = 3\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{6n+3}$$

$$\arctan(x^3) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{2n+1}$$

Note: A much more direct way to work this problem is to recall that $\arctan x = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{2n+1}$, in which case, by substitution,

$$\arctan(x^3) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{2n+1}$$

10. E: To represent $\dfrac{1}{9+4x^2}$ as a power series, note that

$$\frac{1}{9+4x^2} = \frac{1}{9\left(1+\frac{4x^2}{9}\right)} = \frac{1}{9}\frac{1}{1-\left(-\frac{4x^2}{9}\right)}$$

$= \dfrac{1}{9}\sum_{n=0}^{\infty}\left(\dfrac{-4x^2}{9}\right)^n$, where $\left|\dfrac{-4x^2}{9}\right| < 1$.

Thus $|x^2| < \dfrac{9}{4}$, hence $|x| < \dfrac{3}{2}$. Thus the radius of convergence is $R = \dfrac{3}{2}$

11. To find the radius and interval of convergence of the series $\sum_{n=1}^{\infty}\dfrac{(-3)^n(2x-1)^n}{n}$,

use the Ratio Test: $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}(2x-1)^{n+1}}{n+1}\frac{n}{(-3)^n(2x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-3(2x-1)}{n+1}n\right|$$

$= |-3(2x-1)|$

$= |-3||(2x-1)|$

$= 3|2(x-\frac{1}{2})|$

$= 6|x-\frac{1}{2}|$. Now the Ratio Test says if $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges. Hence

$6|x-\frac{1}{2}| < 1$, so $|x-\frac{1}{2}| < \dfrac{1}{6}$, thus $R = \dfrac{1}{6}$. Now test the endpoints of the interval:

$|x-\frac{1}{2}| < \dfrac{1}{6}$ yields $\dfrac{1}{3} < x < \dfrac{2}{3}$. Testing $x = \dfrac{1}{3}$: $\sum_{n=1}^{\infty}\dfrac{(-3)^n(-1/3)^n}{n} = \sum_{n=1}^{\infty}\dfrac{1}{n}$ which diverges by p-series.

Testing $x = \dfrac{2}{3}$: $\sum_{n=1}^{\infty}\dfrac{(-3)^n(1/3)^n}{n} = \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ which converges by the Alternating Series Test. Thus the interval of convergence is $\dfrac{1}{3} < x \leq \dfrac{2}{3}$.

12. (i) To find a Maclaurin Series representaton for $f(x) = \sin\left(\frac{x^2}{3}\right)$, recall that

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Thus

$$f(x) = \sin\left(\frac{x^2}{3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^2}{3}\right)^{2n+1}}{(2n+1)!}.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{3^{2n+1}(2n+1)!}.$$

(ii) $\int_0^1 \sin\left(\frac{x^2}{3}\right)\, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{3^{2n+1}(2n+1)!}\, dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{3^{2n+1}(4n+3)(2n+1)!}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(4n+3)(2n+1)!}$$

(iii) The sum of the first three nonzero terms is

$$s_2 = \frac{1}{3(3)} - \frac{1}{3^3(3!)(7)} + \frac{1}{3^5(5!)(11)}.$$

$$|R_2| < \frac{1}{3^7 7!(15)}$$

13. To determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^4}$ converges absolutely, we will consider the series

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n(\ln n)^4}\right|$$

$= \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$. We will use the integral test with $f(x) = \frac{1}{x(\ln x)^4}$.

$\int_2^{\infty} \frac{1}{x(\ln x)^4}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^4}\, dx$. Using u-substitution with $u = \ln x$, we find

$$\lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^4}\, dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^4}\, du$$

$$= \lim_{t\to\infty} -\frac{1}{3u^3}\Bigg|_{\ln 2}^{\ln t}$$

$= \frac{1}{3(\ln 2)^3}$. Thus since $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n(\ln n)^4}\right|$ converges, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^4}$ converges absolutely.

14. The Taylor series for $f(x) = \ln x$ at $a = 4$ is $\ln x = \sum_{n=0}^{\infty} \frac{f^n(4)}{n!}(x-4)^n$. We must find a formula for $f^n(4)$. To do this, we will try to find a pattern for the deriavtives:

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f^3(x) = \frac{2}{x^3}$

$f^4(x) = -\frac{3*2}{x^4}$

$f^5(x) = \frac{4*3*2}{x^5}$

From this pattern, we can see that

$f^n(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$ for $n > 0$.

Thus $f^n(4) = (-1)^{n+1}\frac{(n-1)!}{4^n}$ for $n > 0$.

$$\ln x = \sum_{n=0}^{\infty} \frac{f^n(4)}{n!}(x-4)^n$$

$$= f(4) + \sum_{n=1}^{\infty} \frac{f^n(4)}{n!}(x-4)^n$$

$$= \ln(4) + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(n-1)!}{4^n n!}(x-4)^n$$

$$= \ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n}(x-4)^n$$

Note: Another acceptable way to work this problem is to note that

$\ln(x) = \ln(4 + (x-4)) = \ln(4) + \ln(1 + (x-4)/4)$ and then to use the Taylor series for $\ln(1+u)$ about $a = 0$ with $u = (x-4)/4$.

15. (i) The fourth degree Taylor polynomial for $f(x)$ centered at $a = 2$ is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3 + \frac{f''''(2)}{24}(x-2)^4.$$

$$= 1 - (x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{24}(x-2)^4$$

(ii) Now, by Taylor's inequality,

$|R_4(x)| < \frac{M}{5!}|x-2|^5$, where $M = max|f^{(5)}(x)|$ for $-1 \le x \le 5$. Now, $|f^5(x)| = |-e^{2-x}|$ which takes on a maximum when $x = -1$. Therefore $M = e^3$.

$$|R_4(x)| < \frac{e^3}{5!}|x-2|^5 \le \frac{e^3}{5!}(3)^5.$$