Fall 2010 Math 152

Exam I Version A Solutions

1. **E** Integrate by parts with $u = x^2$, $dv = e^x\,dx$. Then $\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx$. Integrate by parts again with $u = 2x$, $dv = e^x\,dx$ yields $x^2e^x - 2xe^x + \int 2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$.

2. **E** The force (weight) of the rope and weight after it has been pulled up x feet is $F(x) = 200 - 2x$. Therefore, the work done is given by $W = \int_0^{20} (200 - 2x)\,dx = 200x - x^2\Big|_0^{20}$ $= 4000 - 400 = 3600$ ft-lbs.

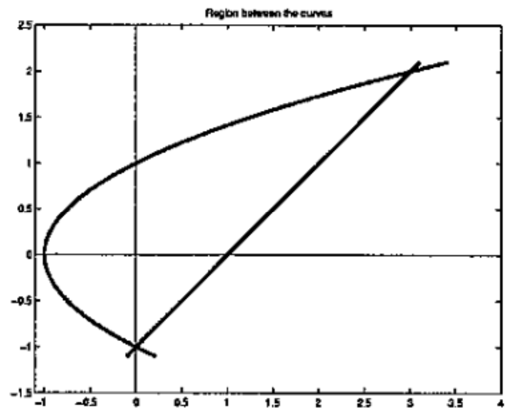
3. **B** Let $u = x^2 + 1$. Then $du = 2x\,dx$. When $x = 0, u = 1$, and when $x = 1, u = 2$. Substituting into the integral yields $\int_1^2 \frac{1}{2}u^{-1/2}\,du = u^{1/2}\Big|_1^2 = \sqrt{2} - 1$.

4. **C** Using the double-angle identity for $\sin(2x)$ yields $\int_0^{\pi/2} 2\sin(x)\cos^2(x)\,dx$. Let $u = \cos(x)$. Then $du = -\sin(x)\,dx$. When $x = 0\,u = 1$, and when $x = \frac{\pi}{2}, u = 0$. Substituting into the integral yields $-\int_1^0 2u^2\,du = \int_0^1 2u^2\,du = \frac{2}{3}u^3\Big|_0^1 = \frac{2}{3}$.

5. **D** Partition along the y-axis. The slices are square prisms $2x \times 2x \times dy = 4x^2\,dy = 4y\,dy$. Therefore $V = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2$.

6. **B** Let $u = 10 + x$. Then $du = dx$. When $x = -1, u = 9$, and when $x = 6, u = 16$. The integral is therefore equivalent to $\int_9^{16} \frac{x}{\sqrt{u}}\,du$. Since $u = 10 + x$, $x = u - 10$, so this integral is equivalent to $\int_9^{16} \frac{u - 10}{\sqrt{u}}\,du = \int_9^{16} (u^{1/2} - 10u^{-1/2})\,du$.

7. **A** The sketch of the region is shown below. Use a partition of the y-axis, so the functions are $x = y + 1$ and $x = y^2 - 1$. These functions intersect when $y + 1 = y^2 - 1$, $y^2 - y - 2 = 0$, $(y - 2)(y + 1) = 0$, or $y = 2, -1$. The area is therefore $\int_{-1}^{2} \Big((y + 1) - (y^2 - 1)\Big)\,dy$

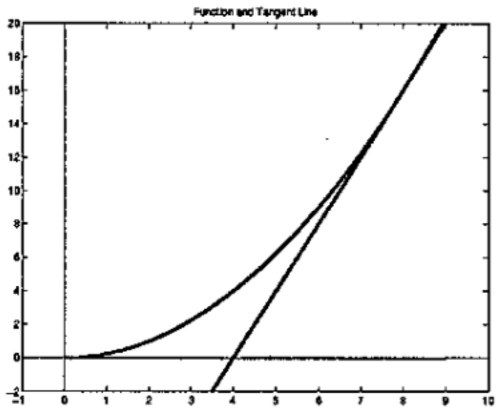


8. **A** Create cylindrical shells centered around $x = 2$. The volume of a shell is $V = 2\pi rh\,dr = 2\pi(2 - x)(x - x^2)\,dx$. Since the curves intersect at $x = 0$ and $x = 1$, the volume of the solid is $V = 2\pi \int_0^1 (2 - x)(x - x^2)\,dx$.

9. **B** Integrate by parts with $u = \ln x$, $dv = x^3\,dx$. Then $du = \frac{1}{x}$ and $v = \frac{1}{4}x^4$. The integral is equivalent to $\frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \left(\frac{1}{x}\right)\,dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3\,dx$.

10. **D** $\sec^4 x = \sec^2 x \cdot \sec^2 x = (\tan^2 x + 1)\sec^2 x$. Let $u = \tan x$. Then $du = \sec^2 x\,dx$. When $x = 0, u = 0$, and when $x = \frac{\pi}{4}, u = 1$. Substituting into the integral yields $\int_0^1 (u^2 + 1)\,du = \frac{1}{3}u^3 + u\Big|_0^1 = \frac{4}{3}$.

11. The graph is shown below. When $x = 8$, $y = 16$ and $f'(x) = 4$, so the equation of the tangent line is $y = 16+4(x-8) = 4x-16$. It is easiest to partition the y-axis, so the functions become $x = 2\sqrt{y}$ and $x = \frac{1}{4}(y+16)$. The area is given by $\int_0^{16} \left(\left(\frac{1}{4}y + 4 \right) - (2\sqrt{y}) \right) dy$. Integrating yields $\frac{1}{8}y^2 + 4y - \frac{4}{3}y^{3/2}\Big|_0^{16} = \left(32 + 64 - \frac{256}{3} \right) = \frac{32}{3}$.



12. .

(a) Let $u = \tan^{-1} x$ and $dv = x\,dx$. Then $du = \frac{1}{x^2+1}\,dx$ and $v = \frac{1}{2}x^2$. Then $\int x \tan^{-1} x\,dx$ $= \frac{1}{2}x^2 \tan^{-1} x - \int \frac{1}{2}x^2 \frac{1}{x^2+1}\,dx$. $= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2(+1-1)}{x^2+1}\,dx =$ $\frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx =$ $\frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C$.

(b) $\int \cos^2 x \tan^3 x\,dx = \int \frac{\sin^3 x}{\cos x}\,dx$ $= \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so the integral is equivalent to $-\int \frac{1-u^2}{u}\,du = -\int \left(\frac{1}{u} - u\right) du =$ $-\ln|u| + \frac{1}{2}u^2 + C = -\ln|\cos x| + \frac{1}{2}\cos^2 x + C$.

13. A cross-sectional slice a distance y from the bottom is cylindrical with radius x and height dy. From similar triangles, $\frac{2}{3} = \frac{x}{y}$, so $x = \frac{2}{3}y$. The force (weight) acting on the slice is $F = \rho g \pi \left(\frac{2}{3}y\right)^2 dy$. The distance required to move the slice out of the tank is $4 - y$ (including the spout), so the work required to pump the water out of the tank is $W = \int_0^3 \rho g \pi \cdot \frac{4}{9}y^2(4-y)\,dy$ $= \frac{4\rho g \pi}{9}\int_0^3 (4y^2 - y^3)\,dy =$ $\frac{4\rho g \pi}{9}\left(\frac{4}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^3 = \frac{4\rho g \pi}{9}\left(36 - \frac{81}{4}\right) =$ $7\rho g \pi$ ft-lbs.

14. Slice perpendicular to the x-axis. The slices are washers with outer radius $\sec x$, inner radius $2\sin x$, and height dx. Therefore, the volume of the solid is $V = \pi\int_0^{\pi/4} (\sec^2 x - 4\sin^2 x)\,dx$ $= \pi\int_0^{\pi/4} \left(\sec^2 x - 2(1 - \cos(2x))\right) dx$ $= \pi\Big(\tan(x) - 2x + \sin(2x)\Big|_0^{\pi/4} =$ $\pi\left(1 - \frac{\pi}{2} + 1\right) = 2\pi - \frac{\pi^2}{2}$.

15. The average value is found by calculating $\frac{1}{b-0}\int_0^b (3x^2 + 4x - 7)\,dx = \frac{1}{b}(x^3 + 2x^2 - 7x\Big|_0^b = b^2 + 2b - 7$. Set this equal to 8 and solve: $b^2 + 2b - 7 = 8$, $b^2 + 2b - 15 = 0$, $(b+5)(b-3) = 0$, so $b = -5$ or $b = 3$. Since $b > 0$, our only solution is $b = 3$.