

# Fall 2010 Math 152

## Exam II Version A Solutions

- C** As  $n \rightarrow \infty$ ,  $e^{-n} \rightarrow 0$ , so  $a_n \rightarrow \frac{1}{5-0} = \frac{1}{5}$ .
- A**  $|a_n| = \frac{2n^2 + 2}{3n^2 + 1} \rightarrow \frac{2}{3}$ , so the terms of the sequence alternate between approaching  $\frac{2}{3}$  (even-numbered terms) and approaching  $-\frac{2}{3}$  (odd-numbered terms), which means the sequence diverges.
- D** The denominator contains a repeating linear factor and an irreducible quadratic factor, so the form of the partial fraction decomposition is  $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+3}$ .
- A** Let  $s$  be the sum of the series. Then  $s = \lim_{n \rightarrow \infty} s_n = 2$ , so statement (I) is true. This means the series is convergent, which makes (II) false. Since the series is convergent,  $a_n \rightarrow 0$  by the Test for Divergence, so (III) is true.
- B** If  $S$  is the surface area,  $S = \int_a^b 2\pi r ds$ . We choose to integrate with respect to  $x$ , so  $r = y = e^{2x}$ .  $\frac{dy}{dx} = 2e^{2x}$ , so  $ds = \sqrt{1 + (2e^{2x})^2} = \sqrt{1 + 4e^{4x}}$ . Therefore,  $S = \int_0^1 2\pi e^{2x} \sqrt{1 + 4e^{4x}} dx$ .
- E** Let  $x = 2 \sin \theta$ . Then  $dx = 2 \cos \theta d\theta$ . When  $x = \sqrt{3}$ ,  $\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ . When  $x = 2$ ,  $\theta = \sin^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{2}$ . Then

$$\int_{\sqrt{3}}^2 \sqrt{4-x^2} dx = \int_{\pi/3}^{\pi/2} \sqrt{4-4\sin^2\theta} (2\cos\theta d\theta)$$

$$= \int_{\pi/3}^{\pi/2} (2\cos\theta)(2\cos\theta) d\theta = 4 \int_{\pi/3}^{\pi/2} \cos^2\theta d\theta.$$
- B** The function is unbounded at  $x = 0$ , so we rewrite the integral as  $\int_{-1}^0 \frac{1}{x^2} dx + \int_0^3 \frac{1}{x^2} dx$ . The Type-2 improper integral  $\int_0^a \frac{1}{x^p} dx$  converges if and only if  $p < 1$ , so both integrals diverge.
- E** Since  $\sin^2 x \leq 1$ ,  $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ . The Type-1 improper integral  $\int_a^\infty \frac{1}{x^p} dx$  converges if and only if  $p > 1$ , so  $\int_1^\infty \frac{\sin^2 x}{x^2} dx$  converges by comparison with  $\int_1^\infty \frac{1}{x^2} dx$ .
- E** The Test for Divergence fails since  $\frac{n}{n^3-5} \rightarrow 0$ . Compare with  $\sum_{n=2}^\infty \frac{1}{n^2}$ , which is convergent by the P-test. Since  $\frac{n}{n^3-5} > \frac{1}{n^2}$ , the Comparison Test fails. Using the Limit Comparison Test, we see that  $\lim_{n \rightarrow \infty} \frac{\frac{n}{n^3-5}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3-5} = 1$ , so the given series is convergent.
- B** (a), (b), and (d) are P-series with  $p = 1, \frac{3}{2}, \frac{1}{3}$  respectively, so only (b) is convergent. Since  $\int_2^\infty \frac{1}{x \ln x} dx = \lim_{a \rightarrow \infty} \ln(\ln x) \Big|_2^a = \infty$ , (c) is divergent by the Integral Test.

11. Let  $x = 2 \sec \theta$ . Then  $dx = 2 \sec \theta \tan \theta d\theta$ . 13.

Substituting into the integral yields

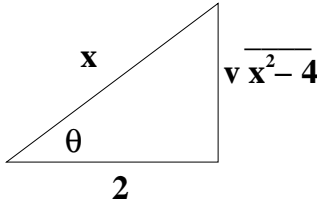
$$\int \frac{2 \sec \theta \tan \theta}{4 \sec^2 \theta \sqrt{4 \sec^2 \theta - 4}} d\theta = \int \frac{2 \sec \theta \tan \theta}{4 \sec^2 \theta (2 \tan \theta)} d\theta$$

$$= \int \frac{1}{4 \sec \theta} d\theta = \frac{1}{4} \int \cos \theta d\theta = \frac{1}{4} \sin \theta + C.$$

Using the reference triangle below,

$$\sin \theta = \frac{\sqrt{x^2 - 4}}{x}, \text{ so the integral}$$

$$= \frac{1}{4} \frac{\sqrt{x^2 - 4}}{x} + C.$$



12. .

(a)  $\sum_{n=0}^{\infty} \frac{2 + 2^n}{10^n} = \sum_{n=0}^{\infty} \frac{2}{10^n} + \sum_{n=0}^{\infty} \frac{2^n}{10^n}$ , assuming both series are convergent. The first is a geometric series with  $a = 2$ ,  $r = \frac{1}{10}$ , and the second is a geometric series with  $a = 1$ ,  $r = \frac{2}{10}$ , so both series are convergent and the total sum is  $\frac{2}{1 - \frac{1}{10}} + \frac{1}{1 - \frac{2}{10}} = \frac{20}{9} + \frac{5}{4} = \frac{125}{36}$ .

(b) Using partial fractions, we find that  $\frac{1}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$ . Therefore, the  $N$ th partial sum of the series is given by  $s_N = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \left(\frac{1}{N} - \frac{1}{N+2}\right) + \left(\frac{1}{N+1} - \frac{1}{N+3}\right) = 1 + \frac{1}{2} - \frac{1}{N+2} - \frac{1}{N+3}$  as a telescoping series. Since  $s = \lim_{N \rightarrow \infty} s_N = 1 + \frac{1}{2}$ , the series converges to  $\frac{3}{2}$ .

(a)  $\frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t$ .  $\frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t$ . Therefore, the length of the curve is given by  $\int_0^{\pi/2} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt = \int_0^{\pi/2} t dt = \frac{1}{2} t^2 \Big|_0^{\pi/2} = \frac{\pi^2}{8}$ .

(b)  $S = \int_a^b 2\pi r ds$ . Since we are rotating about the  $y$ -axis,  $r = x = \cos t + t \sin t$  and, from the previous problem,  $ds = t$ . Therefore, the surface area is given by  $2\pi \int_0^{\pi/2} (\cos t + t \sin t) t dt$

14.  $\frac{3x^2 - 4x + 11}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ . Eliminating the fractions yields  $3x^2 - 4x + 11 = A(x^2 + 4) + (Bx + C)(x - 1)$ . If  $x = 1$ ,  $10 = 5A$ , so  $A = 2$ . Expanding the right-hand side yields  $3x^2 - 4x + 11 = 2x^2 + 8 + Bx^2 - Bx + Cx - C$ . From the  $x^2$  coefficients, we must have  $B = 1$ , and from the constants, we must have  $C = -3$ . Therefore, the given integral is equivalent to  $\int \left(\frac{2}{x-1} + \frac{x-3}{x^2+4}\right) dx = \int \left(\frac{2}{x-1} + \frac{x}{x^2+4} - \frac{3}{x^2+4}\right) dx = 2 \ln|x-1| + \frac{1}{2} \ln|x^2+4| - \frac{3}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$ .

15.

(a) Let  $f(x) = 3x^2 e^{-x^3}$ .  $f$  is continuous, positive, and decreasing ( $f'(x) = 3x e^{-x^3} (2 - 3x^3) < 0$ ), so we can apply the Integral Test.  $\int_1^{\infty} 3x^2 e^{-x^3} dx = \lim_{a \rightarrow \infty} -e^{-x^3} \Big|_1^a = e^{-1}$ , so the integral converges and therefore the given series is convergent by the Integral Test.

(b) By the remainder theorem,  $s - s_3 \leq \int_3^{\infty} 3x^2 e^{-x^3} dx = \lim_{a \rightarrow \infty} -e^{-x^3} \Big|_3^a = e^{-27}$ .