Fall 2010 Math 152

Exam II Version B Solutions

1. **A** As $n \to \infty$, $e^{-n} \to 0$, so $a_n \to \frac{1}{5-0} = \frac{1}{5}$.

2. **B** Since $\sin^2 x \leq 1$, $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$. The Type-1 improper integral $\int_a^\infty \frac{1}{x^p}\,dx$ converges if and only if $p > 1$, so $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges by comparison with $\int_1^\infty \frac{1}{x^2}\,dx$.

3. **E** $|a_n| = \frac{2n^2+2}{3n^2+1} \to \frac{2}{3}$, so the terms of the sequence alternate between approaching $\frac{2}{3}$ (even-numbered terms) and approaching $-\frac{2}{3}$ (odd-numbered terms), which means the sequence diverges.

4. **D** The Test for Divergence fails since $\frac{n}{n^3-5} \to 0$. Compare with $\sum_{n=2}^{\infty} \frac{1}{n^2}$, which is convergent by the P-test. Since $\frac{n}{n^3-5} > \frac{1}{n^2}$, the Comparison Test fails. Using the Limit Comparison Test, we see that $\lim_{n\to\infty} \frac{\frac{n}{n^3-5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3-5} = 1$, so the given series is convergent.

5. **D** (a), (c), and (d) are P-series with $p = 1, \frac{1}{3}, \frac{3}{2}$ respectively, so only (d) is convergent. Since $\int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{a\to\infty} \ln(\ln x)\Big|_2^a = \infty$, (b) is divergent by the Integral Test.

6. **E** The denominator contains a repeating linear factor and an irreducible quadratic factor, so the form of the partial fraction decomposition is $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+3}$.

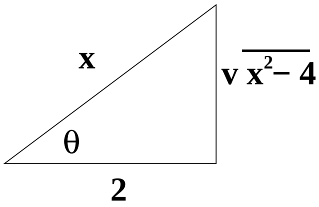
7. **E** The function is unbounded at $x = 0$, so we rewrite the integral as $\int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^3 \frac{1}{x^2}\,dx$. The Type-2 improper integral $\int_0^a \frac{1}{x^p}\,dx$ converges if and only if $p < 1$, so both integrals diverge.

8. **C** If S is the surface area, $S = \int_a^b 2\pi r\,ds$. We choose to integrate with respect to x, so $r = y = e^{2x}$. $\frac{dy}{dx} = 2e^{2x}$, so $ds = \sqrt{1 + (2e^{2x})^2} = \sqrt{1 + 4e^{4x}}$. Therefore, $S = \int_0^1 2\pi e^{2x} \sqrt{1+4e^{4x}}\,dx$.

9. **A** Let s be the sum of the series. Then $s = \lim_{n\to\infty} s_n = 2$, so statement (I) is true. This means the series is convergent, which makes (II) false. Since the series is convergent, $a_n \to 0$ by the Test for Divergence, so (III) is true.

10. **B** Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$. When $x = \sqrt{3}$, $\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. When $x = 2$, $\theta = \sin^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{2}$. Then $\int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx = \int_{\pi/3}^{\pi/2} \sqrt{4 - 4\sin^2\theta}\,(2\cos\theta\,d\theta)$ $= \int_{\pi/3}^{\pi/2} (2\cos\theta)(2\cos\theta)\,d\theta = 4\int_{\pi/3}^{\pi/2} \cos^2\theta\,d\theta$.

11. Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$. Substituting into the integral yields $\int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\,d\theta = \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta(2\tan\theta)}\,d\theta$ $= \int \frac{1}{4\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + C$. Using the reference triangle below, $\sin\theta = \frac{\sqrt{x^2-4}}{x}$, so the integral $= \mathbf{\frac{1}{4}\frac{\sqrt{x^2-4}}{x} + C}$.



12. .

(a) $\sum_{n=0}^{\infty}\frac{2+2^n}{10^n} = \sum_{n=0}^{\infty}\frac{2}{10^n} + \sum_{n=0}^{\infty}\frac{2^n}{10^n}$, assuming both series are convergent. The first is a geometric series with $a = 2, r = \frac{1}{10}$, and the second is a geometric sereis with $a = 1$, $r = \frac{2}{10}$, so both series are convergent and the total sum is $\frac{2}{1-\frac{1}{10}} + \frac{1}{1-\frac{2}{10}} = \frac{20}{9} + \frac{5}{4} = \mathbf{\frac{125}{36}}$.

(b) Using partial fractions, we find that $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$. Therefore, the Nth partial sum of the series is given by $s_N = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+2}\right) + \left(\frac{1}{N+1} - \frac{1}{N+3}\right) = 1 + \frac{1}{2} - \frac{1}{N+2} - \frac{1}{N+3}$ as a telescoping series. Since $s = \lim_{N\to\infty} s_N = 1 + \frac{1}{2}$, the series converges to $\mathbf{\frac{3}{2}}$.

13.

(a) $\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$. $\frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t$. Therefore, the length of the curve is given by $\int_0^{\pi/2}\sqrt{(t\cos t)^2 + (t\sin t)^2}\,dt = \int_0^{\pi/2} t\,dt$ $= \frac{1}{2}t^2\Big|_0^{\pi/2} = \frac{\pi^2}{8}$.

i. $S = \int_a^b 2\pi r\,ds$. Since we are rotating about the y-axis, $r = x = \cos t + t\sin t$ and, from the previous problem, $ds = t$. Therefore, the surface area is given by $\mathbf{2\pi\int_0^{\pi/2}(\cos t + t\sin t)t\,dt}$.

14. $\frac{3x^2-4x+11}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$. Eliminating the fractions yields $3x^2 - 4x + 11 = A(x^2+4) + (Bx+C)(x-1)$. If $x = 1, 10 = 5A$, so $A = 2$. Expanding the right-hand side yields $3x^2 - 4x + 11 = 2x^2 + 8 + Bx^2 - Bx + Cx - C$. From the x^2 coefficients, we must have $B = 1$, and from the constants, we must have $C = -3$. Therefore, the given integral is equivalent to $\int\left(\frac{2}{x-1} + \frac{x-3}{x^2+4}\right)dx$ $= \int\left(\frac{2}{x-1} + \frac{x}{x^2+4} - \frac{3}{x^2+4}\right)dx =$ $\mathbf{2\ln|x-1| + \frac{1}{2}\ln|x^2+4| - \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$.

15.

(a) Let $f(x) = 3x^2e^{-x^3}$. f is continuous, positive, and decreasing ($f'(x) = 3xe^{-x^3}(2 - 3x^3) < 0$), so we can apply the Integral Test. $\int_1^{\infty} 3x^2e^{-x^3}\,dx = \lim_{a\to\infty} -e^{-x^3}\Big|_1^a = e^{-1}$, so the integral converges and therefore the given series is convergent by the Integral Test.

(b) By the remainder theorem, $s - s_3 \leq \int_3^{\infty} 3x^2e^{-x^3}\,dx$ $s - s_3 \leq \lim_{a\to\infty} -e^{-x^3}\Big|_3^a = \mathbf{e^{-27}}$.