Fall 2010 Math 152

Exam III Version A Solutions

1. **E** $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{8}{\sqrt{3}\sqrt{26}} = \dfrac{\mathbf{8}}{\sqrt{\mathbf{78}}}$

2. **E** $f(x)$ is a geometric sum with $a = 1, r = -4x^2$, so $f(x) = \displaystyle\sum_{n=0}^{\infty}(-4x^2)^n = 1 - 4x^2 + 16x^4 \mathbf{-64}x^6 + \cdots$

3. **B** Apply the Ratio Test: $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{4^{n+1}(x-2)^{n+1}}{(n+1)!}\cdot\dfrac{n!}{4^n(x-2)^n}\right| = \dfrac{4|x-2|}{n+1}$, which approaches 0 (hence < 1) for all values of x. Therefore, the radius of convergence is ∞.

4. **C** The series is an Alternating Series, so $|s - s_3| < |a_4| = \mathbf{48e^{-64}}$

5. **D** $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^{-x^2} = \displaystyle\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{\mathbf{n=0}}^{\infty}\frac{\mathbf{(-1)^n x^{2n}}}{\mathbf{n!}}$

6. **B** Apply the Ratio Test to (I): $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+3)3^{n+1}}{2^{2(n+1)+1}}\cdot\dfrac{2^{2n+1}}{(n+2)3^n} = \dfrac{(n+3)3^{n+1}}{2^{2n+3}}\cdot\dfrac{2^{2n+1}}{(n+2)3^n} = \dfrac{3(n+3)}{4(n+2)}$, which approaches $\dfrac{3}{4} < 1$, so **(I) is absolutely convergent**. (II) is an Alternating Series, but $|a_n| = \dfrac{n^3+4}{2n^2} \to \infty$, which means a_n is divergent (hence not approaching 0), so **(II) is divergent by the Test for Divergence**.

7. **A** The power series has a radius of convergence of at least 4, so it is certain that the series is convergent for all $x \in (-4, 4]$ (convergence/divergence is unknown for all other values of x). Only $x = -2$ is in this interval, therefore, we know for certain that $\displaystyle\sum_{\mathbf{n=0}}^{\infty}\mathbf{c_n(-2)^n}$is convergent.

8. **A** $f'(x) = \dfrac{1}{1+x}$, which is a geometric sum with $a = 1, r = -x$, so $f'(x) = \displaystyle\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$. Integration with respect to x yields $f(x) = \displaystyle\sum_{\mathbf{n=0}}^{\infty}\frac{\mathbf{(-1)^n x^{n+1}}}{\mathbf{n+1}}$ (since $f(0) = 0$, the constant $C = 0$).

9. **B** Completing the square yields $(x+1)^2 + (y-2)^2 + z^2 = 5$, so the center is $(\mathbf{-1, 2, 0})$ and the radius is $\sqrt{\mathbf{5}}$.

10. **D** We need $\mathbf{a}\cdot\mathbf{b} = 0$. $(t+2)(t-2) + (t)(t+1) + t^2 = 3t^2 + t - 4 = 0$, $(3t+4)(t-1) = 0$, so $\mathbf{t = -\dfrac{4}{3}}$ **or** $\mathbf{t = 1}$.

11. From the chart below, $\cos x = \dfrac{-(x-\frac{\pi}{2})}{1!} + \dfrac{(x-\frac{\pi}{2})^3}{3!} - \dfrac{(x-\frac{\pi}{2})^5}{5!} + \cdots = \displaystyle\sum_{\mathbf{n=0}}^{\infty}\frac{\mathbf{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}}}{\mathbf{(2n+1)!}}$

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{2}\right)$	$\frac{f^{(n)}(\frac{\pi}{2})}{n!}\left(x-\frac{\pi}{2}\right)^n$
0	$\cos x$	0	0
1	$-\sin x$	-1	$\frac{-1}{1!}\left(x-\frac{\pi}{2}\right)^1$
2	$-\cos x$	0	0
3	$\sin x$	1	$\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3$
4	$\cos x$	0	0

12. .

(a) From the chart below, $\sqrt{x} \approx T_3(x) = \mathbf{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}(x-1)^n$
0	$\sqrt{x}$	1	$\frac{1}{0!}(x-1)^0 = 1$
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$	$\frac{\frac{1}{2}}{1!}(x-1)^1 = \frac{1}{2}(x-1)$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$	$\frac{-\frac{1}{4}}{2!}(x-1)^2 = -\frac{1}{8}(x-1)^2$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$	$\frac{\frac{3}{8}}{3!}(x-1)^3 = \frac{1}{16}(x-1)^3$

(b) $a = 1, n = 3$. To find M, note that $|f^{(4)}(x)| = \left|-\frac{15}{16}x^{-7/2}\right| = \frac{15}{16x^{7/2}}$, which is largest on the interval $x \in [1,2]$ at $x = 1$. Therefore, $M = f^{(4)}(1) = \frac{15}{16}$. Therefore, $|R_3(x)| \leq \frac{\frac{15}{16}}{4!}|x-1|^3$. This error is largest on the interval $x \in [1,2]$ at $x = 2$, so on the interval, $|R_3(x)| \leq \frac{\frac{15}{16}}{4!}|2-1|^3 = \mathbf{\frac{15}{16 \cdot 24}}$.

13.

(a) The series is an Alternating Series. $|a_n| = \frac{n}{n^2+4}$, which is decreasing and approaching 0. Therefore, the series is convergent by the Alternating Series Test.

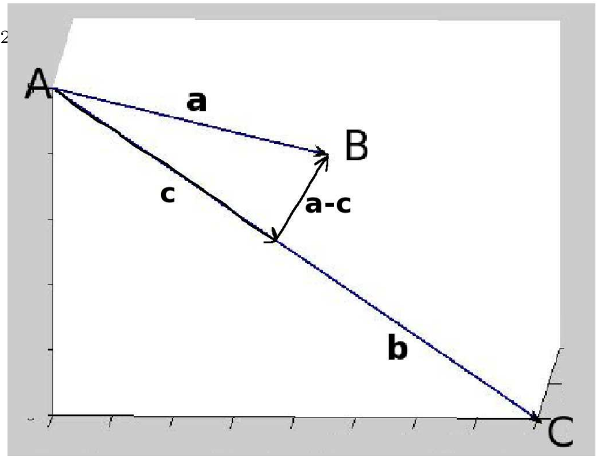
(b) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+4}$. Let $b_n = \frac{1}{n}$. Both $|a_n|$ and b_n are greater than 0, and $\frac{a_n}{b_n} = \frac{n}{n^2+4} \cdot \frac{n}{1} = \frac{n^2}{n^2+4}$, which approaches 1. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the P-Test, $\sum_{n=1}^{\infty} |a_n|$ is divergent by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$, hence the Alternating Series is not absolutely convergent. (NOTE: Integral Test could be used to show divergence as well).

14.

(a)

i. $\mathbf{a} = \langle 3,5,3\rangle - \langle 2,4,5\rangle = \langle \mathbf{1}, \mathbf{1}, \mathbf{-2}\rangle$
ii. $\mathbf{b} = \langle 2,8,-3\rangle - \langle 2,4,5\rangle = \langle \mathbf{0}, \mathbf{4}, \mathbf{-8}\rangle$
iii. $\mathbf{c} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{20}{16+64}\langle 0,4,-8\rangle = \langle \mathbf{0}, \mathbf{1}, \mathbf{-2}\rangle$
iv. $\mathbf{a} - \mathbf{c} = \langle \mathbf{1}, \mathbf{0}, \mathbf{0}\rangle$

(b)



15. Apply the Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+3)^{n+1}}{(n+3)4^{n+1}} \cdot \frac{(n+2)4^n}{(x+3)^n}\right| = \frac{(n+2)}{4(n+3)}|x+3|$. We want the limit, $\frac{|x+3|}{4} < 1$, which is true when $|x+3| < 4$, $-4 < x+3 < 4$, or $-7 < x < 1$. Therefore, the radius of convergence is 4. The series diverges (limit > 1) when $|x+3| > 4$, i.e., $x > 1$ or $x < -7$. Since the Ratio Test is inconclusive (limit $= 1$) at the endpoints, we must test them separately: When $x = -7$, $\sum_{n=1}^{\infty} \frac{(-7+3)^n}{(n+2)4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$, which is convergent by the Alternating Series Test. When $x = 1$, $\sum_{n=1}^{\infty} \frac{(1+3)^n}{(n+2)4^n} = \sum_{n=1}^{\infty} \frac{1}{n+2}$, which is divergent by the Integral Test or by Limit Comparison Test with $\sum \frac{1}{n}$. Therefore, the interval of convergence is $[\mathbf{-7}, \mathbf{1})$.