

PRINT

LAST NAME Linhart FIRST NAME Jean Marie  
 INSTRUCTOR: \_\_\_\_\_ SECTION NUMBER: 501-503  
 UIN: \_\_\_\_\_ SEAT NUMBER: \_\_\_\_\_

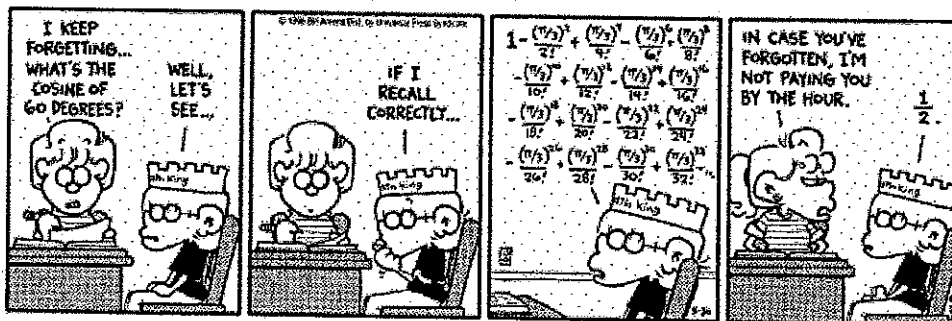
Directions

1. The use of all electronic devices is prohibited.
2. In Part 1 (Problems 1-12), mark the correct choice on your Scantron using a No. 2 pencil. Record your choices on your exam. Scantrons will not be returned.
3. In Part 2 (Problems 13-17), present your solutions in the space provided. Show all your work neatly and concisely and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
4. Be sure to write your name, section and version letter of the exam on the Scantron form.
5. Any scratch paper used must be handed in with the exam.
6. Good Luck!

THE AGGIE CODE OF HONOR

“An Aggie does not lie, cheat, or steal, or tolerate those who do.”

Signature: Jean Marie Linhart



<http://foxtrot.com>

| Question        | 1-12 | 13 | 14 | 15 | 16 | 17 | TOTAL |
|-----------------|------|----|----|----|----|----|-------|
| Points Awarded  | 48   | 7  | 11 | 14 | 10 | 10 | 100   |
| Points Possible | 48   | 7  | 11 | 14 | 10 | 10 | 100   |

1. The equation of the sphere passing through the point (2, 1, 3) with center (1, -2, 3) is

- (a)  $(x - 2)^2 + (y + 2)^2 + (z - 3)^2 = 100$
- (b)  $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 10$
- (c)  $(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10$
- (d)  $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = \sqrt{10}$
- (e)  $(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = \sqrt{10}$

radius =  $\sqrt{(2-1)^2 + (1+2)^2 + (3-3)^2} = \sqrt{10}$

So

$(x-1)^2 + (y+2)^2 + (z-3)^2 = 10$

2. What is the cosine of the angle between the vectors  $\langle -1, 2, -4 \rangle$  and  $\langle 4, -2, -1 \rangle$ ?

- (a)  $\frac{12}{\sqrt{21}}$
- (b)  $\frac{-12}{\sqrt{21}}$
- (c)  $\frac{-12}{21}$
- (d)  $\frac{4}{\sqrt{21}}$
- (e)  $\frac{-4}{21}$

$\langle -1, 2, -4 \rangle \cdot \langle 4, -2, -1 \rangle = \|\langle -1, 2, -4 \rangle\| \|\langle 4, -2, -1 \rangle\| \cos \theta$

$\cos \theta = \frac{-4 - 4 + 4}{\sqrt{1+4+16} \sqrt{16+4+1}} = \frac{-4}{21}$

3. What is the distance between points  $(-3, -2, 4)$  and  $(1, -3, -4)$ ?

- (a)  $\sqrt{33}$
- (b)  $\sqrt{73}$
- (c) 9
- (d) 13
- (e) 81

$d = \sqrt{(-3-1)^2 + (-2+3)^2 + (4+4)^2}$   
 $= \sqrt{16+1+64} = \sqrt{81} = 9$

4. What is the power series representation of  $f(x) = \frac{1}{4-9x^2}$  at  $x = 0$ ?

- (a)  $\sum_{n=0}^{\infty} \frac{4^n x^{2n}}{9^{n+1}}$
- (b)  $\sum_{n=0}^{\infty} \frac{9^n x^{2n}}{4^{n+1}}$
- (c)  $\sum_{n=0}^{\infty} (-1)^n \frac{9^n x^{2n}}{4^{n+1}}$
- (d)  $\sum_{n=0}^{\infty} (-1)^n \frac{4^n x^{2n}}{9^{n+1}}$
- (e)  $\sum_{n=1}^{\infty} \frac{9^n x^{2n}}{4}$

$f(x) = \frac{1}{4} \left( \frac{1}{1 - \frac{9}{4}x^2} \right) = \frac{1/4}{1 - \frac{9}{4}x^2}$

$a = 1/4 \quad r = \frac{9}{4}x^2$

$\sum_{n=0}^{\infty} \frac{1}{4} \left( \frac{9}{4}x^2 \right)^n = \sum_{n=0}^{\infty} \frac{9^n x^{2n}}{4^{n+1}}$

5. Given that the power series  $\sum_{n=0}^{\infty} c_n x^n$  converges when  $x = 4$  and diverges when  $x = 6$ , which of the statements is certain to be true?

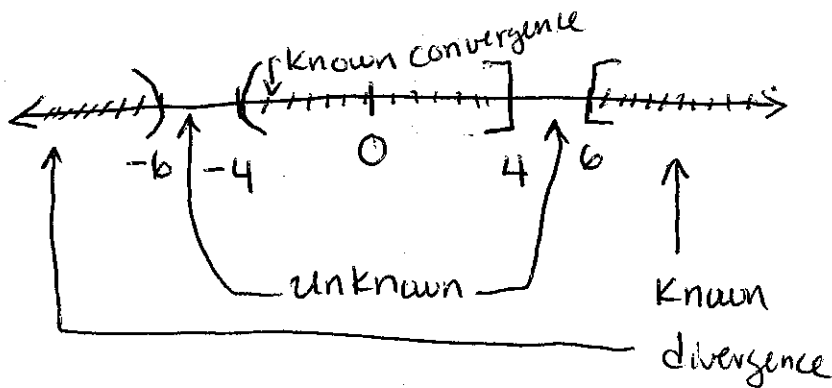
(a)  $\sum_{n=0}^{\infty} c_n (-3)^n$  is convergent

(b)  $\sum_{n=0}^{\infty} c_n (5)^n$  is divergent

(c)  $\sum_{n=0}^{\infty} c_n (-5)^n$  is convergent

(d)  $\sum_{n=0}^{\infty} c_n (-6)^n$  is divergent

(e) None of these statements is certain to be true.



6. What is the Maclaurin series for  $e^{2x^2}$ ?

(a)  $\sum_{n=0}^{\infty} \frac{2^n x^n}{(n!)}$

(b)  $\sum_{n=0}^{\infty} \frac{2x^{n+2}}{(n!)}$

(c)  $\sum_{n=0}^{\infty} \frac{2x^n}{(n!)}$

(d)  $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{(n!)}$

(e)  $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{(2n!)}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{2x^2} = \sum_{n=0}^{\infty} \frac{(2x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$$

7. Which series converges absolutely?

(a)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}}$

← p-series  $p < 1$  diverges with  $\sum$  absolute value

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^n e^{2n}}{e^{2n+1}}$

←  $\sum \frac{(-1)^n}{e}$  diverges by Divergence Test

(c)  $\sum_{n=1}^{\infty} (\sqrt{2n} - \sqrt{n})$

← Diverges

(d)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

←  $\sum \frac{1}{n}$  diverges

(e)  $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

← converges by Ratio Test

8. Which series diverges?

(a)  $\sum_{n=2}^{\infty} \frac{2n^2 - n + 2}{3n^3 + 4}$

← Limit comparison with  $\sum \frac{1}{n}$  (diverges)

(b)  $\sum_{n=0}^{\infty} \frac{1}{n^2 + 4n + 6}$

$\lim_{n \rightarrow \infty} \frac{2n^2 - n + 2}{3n^3 + 4} \cdot \frac{n}{1} =$

(c)  $\sum_{n=1}^{\infty} 2ne^{-n^2}$

(d)  $\sum_{n=1}^{\infty} \frac{1}{n!}$

$\lim_{n \rightarrow \infty} \frac{2n^3 - n^2 + 2n}{3n^3 + 4} =$

(e)  $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$

Lead Order term

$\frac{2}{3}$   
so both series diverge

9. Let  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  be nonzero vectors where  $\|\mathbf{a}\|$  is the length of  $\mathbf{a}$ . Given

$0 < |\mathbf{a} \cdot \mathbf{b}| < \|\mathbf{a}\|\|\mathbf{b}\| \Rightarrow 0 < \cos \theta < 1$

and

$\Rightarrow$  neither

$\mathbf{c} \cdot \mathbf{d} = \|\mathbf{c}\|\|\mathbf{d}\| \Rightarrow \cos \theta = 1 \Rightarrow$  parallel

$\theta = 0$

Which of these statements is true?

- (a)  $\mathbf{a}$  and  $\mathbf{b}$  are parallel;  $\mathbf{c}$  and  $\mathbf{d}$  are neither orthogonal nor parallel
- (b)  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal;  $\mathbf{c}$  and  $\mathbf{d}$  are neither orthogonal nor parallel
- (c)  $\mathbf{a}$  and  $\mathbf{b}$  are neither orthogonal nor parallel;  $\mathbf{c}$  and  $\mathbf{d}$  are parallel.
- (d)  $\mathbf{a}$  and  $\mathbf{b}$  are neither orthogonal nor parallel;  $\mathbf{c}$  and  $\mathbf{d}$  are orthogonal.
- (e) None of the above statements is true.

10. Find a unit vector in the direction of the vector  $\langle -1, -2, 2 \rangle$ .

(a)  $\frac{1}{9} \langle -1, -2, 2 \rangle$

(b)  $\frac{1}{9} \langle 1, 2, -2 \rangle$

(c)  $\frac{1}{3} \langle -1, -2, 2 \rangle$

(d)  $3 \langle -1, -2, 2 \rangle$

(e)  $9 \langle -1, -2, 2 \rangle$

$\frac{\langle -1, -2, 2 \rangle}{\sqrt{(-1)^2 + (-2)^2 + 2^2}} = \frac{\langle -1, -2, 2 \rangle}{\sqrt{9}} =$

$\frac{1}{3} \langle -1, -2, 2 \rangle$

11. Find the vector projection of the vector  $\langle 1, 2, 3 \rangle$  onto the vector  $\langle 3, 2, 1 \rangle$ .

(a)  $\frac{5}{7}$  ← not a vector so cannot be correct

(b)  $\frac{10}{\sqrt{14}}$  ↙

(c)  $\frac{1}{14} \langle 10, 20, 30 \rangle$

(d)  $\frac{1}{14} \langle 30, 20, 10 \rangle$

(e)  $\frac{1}{\sqrt{14}} \langle 10, 20, 30 \rangle$

in direction  $\langle 1, 2, 3 \rangle$  so cannot be correct.

12. Find the radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{5^n (x-3)^n}{n!}$ .

(a)  $\infty$

(b) 5

(c)  $\frac{4}{5}$

(d)  $\frac{1}{5}$

(e) 0

Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{5^{n+1} |x-3|^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n |x-3|^n}$$

$$= \lim_{n \rightarrow \infty} \frac{5 |x-3|}{n+1} = 0 < 1 \text{ always}$$

so  $R = \infty$

## PART II WORK OUT

**Directions:** Present your solutions in the space provided. Show all your work neatly and concisely and box your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

13. (7 points) Compute the Maclaurin series for  $\frac{\sin(x) - x}{x^3}$ .

Maclaurin series for  $\sin(x)$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$\Rightarrow$  Maclaurin Series for  $\sin(x) - x$  is

$$\sin(x) - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Now divide each term by  $x^3$ :

$$\frac{\sin x - x}{x^3} = \frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \frac{x^6}{9!} - \dots = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n+1)!}}$$

14. (6 points) The series  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{(5^n)(2n^2 - n)}$  converges conditionally. Determine whether this series also converges absolutely. Clearly explain your reasoning.

Notice the  $5^n$  in the denominator. This suggests using the Ratio Test, which tests for absolute convergence.

Notice also  $a_n = \frac{(-1)^n n}{5^n (2n^2 - n)} = \frac{(-1)^n}{5^n (2n-1)}$

Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{5^{n+1} (2(n+1))} \cdot \frac{5^n (2n-1)}{1} =$$

$$\lim_{n \rightarrow \infty} \frac{2n-1}{5(2n+1)} = \frac{1}{5} < 1 \quad \text{so by the Ratio Test}$$

this series converges absolutely.

- (5 points) What is a bound on the error if we sum the first 3 terms of the series?

Alternating Series error  $S = \sum_{n=1}^{\infty} (-1)^n a_n$  (total sum)

Then  $|\text{error}| = |S - S_n| < |a_{n+1}|$   $\leftarrow$   $n+1$  term  
 $\uparrow$   $n$ th partial sum

In this case  $|\text{error}| = |S - S_3| < |a_4| = \frac{1}{5^4 (2(4)-1)}$

$$= \boxed{\frac{1}{5^4 \cdot 7}}$$

15. (8 points) Find the first four terms of the Taylor series for  $f(x) = x^{\frac{3}{2}}$  centered at  $a = 9$ .

| $f^{(n)}(x)$              | $f^{(n)}(a)$  |
|---------------------------|---|
| 0) $x^{3/2}$              | $(\sqrt{9})^3 = 27$                                     |
| 1) $\frac{3}{2}x^{1/2}$   | $\frac{3}{2}(3) = \frac{9}{2}$                          |
| 2) $\frac{3}{4}x^{-1/2}$  | $\frac{3}{4}\left(\frac{1}{3}\right) = \frac{1}{4}$     |
| 3) $-\frac{3}{8}x^{-3/2}$ | $-\frac{3}{8}\left(\frac{1}{27}\right) = -\frac{1}{72}$ |

Taylor Series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

4 terms:

$$27 + \frac{9}{2}(x-9) + \frac{1}{4(2!)}(x-9)^2 + \frac{(-1)}{72(3!)}(x-9)^3$$

$$= 27 + \frac{9}{2}(x-9) + \frac{1}{8}(x-9)^2 - \frac{1}{72(6)}(x-9)^3$$

first 4 terms

(6 points) Use Taylor's Inequality to give a bound for the error when using  $T_1(x)$  (the first degree Taylor polynomial) centered at  $a = 9$  to approximate  $f(x) = x^{\frac{3}{2}}$  on  $[8, 10]$ .

Taylor's Inequality:  $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$  on the interval  $[8, 10]$  where  $M = \max |f^{(n+1)}(x)|$  on  $[8, 10]$ .

$$\text{Find } |f(x) - T_1(x)| = |R_1(x)| \Rightarrow n=1$$

$$\text{Need } f^{(2)}(x) = \frac{3}{4}x^{-1/2} \text{ maximized on } [8, 10]$$

$$\max_{[8, 10]} f^{(2)}(x) = \frac{3}{4\sqrt{8}} = \frac{3}{8\sqrt{2}} \quad (\text{at } x=8)$$

$$\text{So } |R_1(x)| \leq \frac{3}{(8\sqrt{2})(2!)} |x-9|^2 = \frac{3}{16\sqrt{2}} |x-9|^2$$

both correct

$$\leq \frac{3}{16\sqrt{2}} \text{ since } |x-9| \leq 1$$

16. (10 points) Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n(n^2+2)}$$

To find the interval of convergence, use the Ratio Test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x-3|^{n+1}}{5^{n+1}[(n+1)^2+2]} \cdot \frac{5^n[n^2+2]}{|x-3|^n} =$$

$$\lim_{n \rightarrow \infty} \frac{|x-3| [n^2+2]}{5 [n^2+2n+3]} = \frac{|x-3|}{5} < 1$$

$$\text{so } |x-3| < 5$$

5 = Radius of conv.

Center = 3 so endpoints are 3-5 and 3+5  
-2 and 8

$$\text{Check } -2: \sum_{n=0}^{\infty} \frac{(-5)^n}{5^n(n^2+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+2}$$

This converges by the Alternating Series Test

$$\lim_{n \rightarrow \infty} \frac{1}{n^2+2} = 0.$$

$$\text{Check } 8: \sum_{n=0}^{\infty} \frac{5^n}{5^n(n^2+2)} = \sum_{n=0}^{\infty} \frac{1}{n^2+2}$$

This converges by the Limit Comparison Test

with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  (converges)  $\lim_{n \rightarrow \infty} \frac{n^2}{n^2+2} = 1$  so both converge.

$$\text{Interval: } [-2, 8]$$

17. (10 points) Find the radius and center of the sphere

$$x^2 + y^2 + z^2 + 2x - 4y + 6z = 11$$

Complete the Squares:

$$x^2 + 2x + \underline{1} + y^2 - 4y + \underline{4} + z^2 + 6z + \underline{9} =$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$a=1, a^2=1$$

$$(y+b)^2 =$$

$$y^2 + 2by + b^2$$

$$b=-2, b^2=4$$

$$(z+c)^2 = z^2 + 2cz + c^2 \quad c=3$$

$$11 + \underline{1+4+9}$$

so

$$(x+1)^2 + (y-2)^2 + (z+3)^2 = 25$$

Center:  $(-1, 2, -3)$

Radius: 5