

Common Exam 2 A. Solutions

1. $\int \frac{x^3}{x+1} dx = \int x^2 - x + 1 - \frac{1}{x+1} dx = \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C$

Answer. B

2. Which of the following integrals is an improper integral?

Answer. C

3. $\int_2^\infty \frac{1}{x(\ln x)^2} dx = -\frac{1}{\ln x} \Big|_2^\infty = \frac{1}{\ln 2}$

Answer. C

4. The masses $m_1 = 3$ and $m_2 = 2$ are located at the points $P_1(-1, 1)$ and $P_2(3, 1)$ respectively.

Since $\bar{x} = \frac{3(-1) + 2(3)}{2 + 3} = \frac{3}{5}$ and $\bar{y} = \frac{3(1) + 2(1)}{2 + 3} = 1$, the center of mass of the system

is $(\frac{3}{5}, 1)$.

Answer. E

5. The length of the parametric curve $x = \cos 2t$, $y = \sin 2t$, $\frac{\pi}{4} \leq t \leq \frac{\pi}{3}$, is equal to

$$\int_{\pi/4}^{\pi/3} \sqrt{(2 \sin 2t)^2 + (2 \cos 2t)^2} dt = \int_{\pi/4}^{\pi/3} 2 dt = \frac{\pi}{6}$$

Answer. D

6. The curve $y = x^2 + 1$, $-1 \leq x \leq 1$ is revolved about the x -axis.

The surface area obtained is given by the integral

$$\int 2\pi y ds = \int_{-1}^1 2\pi(x^2 + 1)\sqrt{1 + 4x^2} dx$$

Answer. E

7. The velocity $v(t)$ (in feet per second) of a moving object is given below at one second increments. Use Simpson's Rule with $n = 4$ to approximate the total distance travelled from $t = 0$ to $t = 4$ seconds.

t	0	1	2	3	4
v(t)	3	2	1	-1	0

Distance travelled is equal to

$$\int_0^4 v(t) dt \approx \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) = \frac{1}{3} (3 + 4(2) + 2(1) + 4(-1) + 0) = 3$$

Answer. A.

8. Consider the integral $\int_0^1 e^{x^2} dx$. There is no elementary antiderivative, so the integral is approximated.

Which of the approximate integration formulas underestimates the integral value?

Answer. B. (Left Point Formula)

9. A curve passes through the point $(0, 1)$ and has the slope at any point (x, y) equal to y^2 . Which of the following points is on the curve?

A) $(2, \frac{1}{2})$ B) $(2, \frac{1}{3})$ C) $(2, -1)$ D) $(2, 1)$ E) $(2, e)$

Solution. The curve is described by the initial value problem $\frac{dy}{dx} = y^2, y(0) = 1$. Separate the variables and integrate to get

$$\frac{dy}{y^2} = dx, \frac{-1}{y} = x + C$$

Use the initial condition $y(0) = 1$ to obtain $C = -1$. Thus, $\frac{-1}{y} = x - 1 \Rightarrow y = \frac{-1}{x-1}$, and hence, $y(2) = -1$.

Answer. C.

10. Which of the following is the direction field for the equation $\frac{dy}{dx} = -\frac{x}{y}$?

Answer. C.

11. Solve the initial value problem $\frac{dy}{dx} - \frac{1}{x}y = xe^{-x}, y(1) = -\frac{1}{e}$.

a) An integrating factor is $\mu(x) = e^{\int p(x) dx} = e^{-\ln x} = \frac{1}{x}$.

b) Multiplying both sides by $\mu(x) = \frac{1}{x}$ and integrating $(\mu y)' = e^{-x}$, we get

$$\frac{1}{x}y = -e^{-x} + C, y = -xe^{-x} + Cx$$

This is the general solution of the equation.

c) The particular solution satisfying the given initial condition, $y(1) = -\frac{1}{e}$, is $y = -xe^{-x}$, since $C = 0$.

d) $\lim_{x \rightarrow \infty} -xe^{-x} = \lim_{x \rightarrow \infty} -\frac{x}{e^x} = 0$ (use L'Hospital's Rule).

12. Find the centroid of the region bounded by $y = \cos 2x, y = 0, x = 0, x = \frac{\pi}{4}$.

a) The area of the region = $A = \int_0^{\pi/4} \cos 2x dx = \frac{1}{2}$.

b) $\bar{x} = \frac{1}{A} \int_0^{\pi/4} x \cos 2x dx = \frac{\pi}{4} - \frac{1}{2}$

(use integration by parts with $u = x, dv = \cos 2x dx, du = dx, v = \frac{1}{2} \sin 2x$).

c) $\bar{y} = \frac{1}{A} \int_0^{\pi/4} \frac{1}{2} (\cos 2x)^2 dx = \pi/8$

(use the power reduction formula = half angle formula, $\cos 2x = \frac{1 + \cos 4x}{2}$)

13. Consider the function $f(x) = \frac{1}{x^4 + x^2}$

a) The partial fraction decomposition of the function $f(x) = \frac{1}{x^4 + x^2} = \frac{1}{x^2(x^2 + 1)}$

has the form $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$,

where $A = C = 0, B = 1, D = -1$.

$$\text{b) } \int \frac{1}{x^4 + x^2} dx = \int \frac{1}{x^2} - \frac{1}{x^2 + 1} dx = -\frac{1}{x} - \arctan x + C$$

c) Since $\int_1^\infty \frac{1}{x^4 + x^2} dx = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} - \arctan t\right) - \left(-1 - \frac{\pi}{2}\right) = (0 - \frac{\pi}{2}) - (-1 - \frac{\pi}{4}) = 1 - \frac{\pi}{4}$, the integral converges.

You can also use a comparison with $\int_1^\infty \frac{1}{x^4} dx$ or with $\int_1^\infty \frac{1}{x^2} dx$ to make the conclusion about the convergence. But here you were asked also to compute the value of the integral, so just the Comparison Theorem is not enough.

14. A plate of a triangular shape is submerged vertically in water so that its base lies at the surface of the water. The plate's base is 4 meters, its height is 3 meters.

Find the hydrostatic force on one side of the plate. (Use the facts that the density of water is $\rho = 1000 \text{ kg/m}^3$ and that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.)

Solution. Consider the x -axis going down, with the origin at the surface of water. Then the depth of the slice is x , the length of the slice is $y = \frac{4}{3}(3 - x)$ (use the similarity of the triangles), the area is $y = \frac{4}{3}(3 - x) dx$ and the hydrostatic force is $\int_0^3 \rho g x \frac{4}{3}(3 - x) dx = 6\rho g$.

Comment: If you use a different coordinate system, your integral may be different, but the answer should be the same.

15. Newton's law of cooling says that the rate of change of a body's temperature is proportional to the difference between the temperature of the body and the temperature of the surrounding air.

A room is kept at 20° C . A boiling pot of soup was brought into the room, and it has cooled from 100° C to 60° C in 10 minutes.

Formulate the initial value problem for $T(t)$, the temperature of the pot at time t .

Solution. Since the temperature of the pot is proportional to the difference between the temperature of the pot and the temperature of the surrounding air, the differential equation for $T(t)$ is $\frac{dT}{dt} = k(T - 20)$ (actually, $k < 0$, so you may wish to write the equation as $\frac{dT}{dt} = -k(T - 20)$, $k > 0$).

The initial condition is $T(0) = 100$ (the pot is boiling, so the initial temperature is 100°).